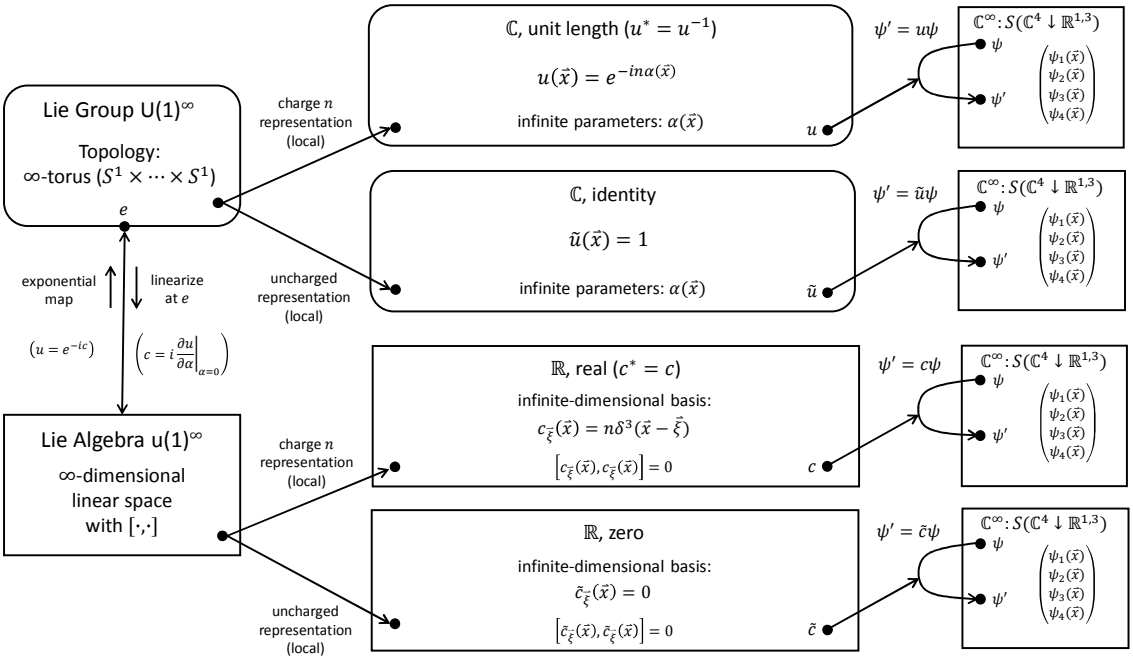
6.8 U(1)$^\infty$: Local Gauge Transformations of the Electron Field; Fiber Bundles



In the previous example, we applied a global phase transformation to the electron field and found that it leaves the Dirac Lagrangian density invariant. In other words, we get different mathematical descriptions for different orientations of the "electron-phase frame" (which marks the zero phase), but they are all equivalent (= describe the same physical reality) and are related by U(1) transformations. We say that the electron field is symmetric under a *global gauge transformation*.

While it is not surprising that the absolute phase of the electron field is irrelevant, we would expect that the *difference* between two phase values, one at event $\vec{x}_a = (t_a, x_a, y_a, z_a)^T$ and one at event $\vec{x}_b = (t_b, x_b, y_b, z_b)^T$, is relevant. Surprisingly, this difference is not even well defined! In general, it depends on the space-time path chosen to connect the two events, that is, the phase difference is *path dependent*. This path dependence is exemplified by the *Aharonov-Bohm effect*, which relies on interference to probe the phase difference between two electron beams (of equal wavelength) propagating from point A to point B along two different paths (of equal length). In the absence of electromagnetism, the interference is constructive revealing equal phases for both paths, but when a magnetic field is applied *between* the two paths (without the field being present at the paths themselves), the interference pattern changes, demonstrating a path-dependent phase [RtR, Ch. 19.4].

How can we describe such a path-dependent phase mathematically? In general, path dependence arises in curved spaces. For example, imagine a sphere, S^2, with two tangent vectors at two different locations: $\vec{v}_a(\vec{x}_a)$ and $\vec{v}_b(\vec{x}_b)$. What is the angle between $\vec{v}_a$ and $\vec{v}_b$? To find out, we first have to *parallel transport* one vector to the other; only then can we measure the angle between the two. But this angle depends on the path that we chose to parallel transport one vector to the other [RtR, Ch. 14.2]! It is the sphere's *curvature* that causes this path dependence; it does not occur on a flat plane. We may be tempted to express the tangent vectors by an ordinary function of position on the sphere, $\vec{v}(\vec{x})$, but this doesn't work. To allow for path dependence, we need the more general notion of a *section* of a *fiber bundle*, specifically, a section of the *tangent bundle* of S^2 [RtR, Ch. 15.7]. This means

that at every point on the sphere, $\vec{x} \in S^2$, there exists an (a priori) independent tangent plane, $\mathbb{R}^2$, and the tangent vector at that point lives in this plane, $\vec{v} \in \mathbb{R}^2$. These tangent planes represent the fibers of the fiber bundle. In addition, we need a rule that tells us for any point on the sphere how to parallel transport a tangent vector at that point to a neighboring point. Such a rule is known as a *connection* because it connects the individual fiber spaces to each other. Together, the fibration and the connection provide the necessary flexibility to describe vectors in arbitrarily curved spaces. This is the idea of Riemannian geometry in a nutshell! See the Appendix "Metric, Connection, and Curvature in 2D Riemannian Geometry" for concrete examples.

Now, let's apply these ideas to our electron field. To permit a path-dependent phase difference, we upgrade $\psi(\vec{x})$ from an ordinary function to a section of a $\mathbb{C}^4$ bundle over $\mathbb{R}^{1,3}$. This means that at every event, $\vec{x} \in \mathbb{R}^{1,3}$, there exists an (a priori) independent fiber space, $\mathbb{C}^4$, and the value of the electron field at that event lives in this space, $\psi \in \mathbb{C}^4$. We can think of each fiber as having its own *local* "electron-phase frame" (which marks the zero phase). In contrast to our geometry example from above, the $\mathbb{C}^4$ fiber space is *not* a tangent space but a field space, a.k.a. an *internal space*. Next, we need to define a connection field on $\mathbb{R}^{1,3}$ that specifies how to "parallel transport" an electron-field value from one fiber to the next. This aspect will be discussed in detail in the next example. While our new description is more general and powerful (allowing for path-dependent phase differences), it is also highly redundant: a huge (actually, infinite) number of mathematical descriptions model the same physical reality. This may seem awkward, but it leads to important constraints, as we will see later! The transformations that relate all these equivalent descriptions are called *local gauge transformations*.

The upper branch of the diagram shows the group of local U(1) gauge transformations acting on the electron field: $\psi'(\vec{x}) = e^{-in\alpha(\vec{x})}\psi(\vec{x})$. This field is given by a section of a $\mathbb{C}^4$ bundle over $\mathbb{R}^{1,3}$, abbreviated as $S(\mathbb{C}^4 \downarrow \mathbb{R}^{1,3})$ in the diagram. The group consists of a separate copy of U(1) at every event, $\vec{x} \in \mathbb{R}^{1,3}$, that is, one copy of U(1) per fiber. Therefore, the total group is U(1)×U(1)× … ×U(1), which we can summarize as $\mathrm{U}(1)^{\infty}$. This group has infinitely many parameters, namely an α parameter at every event, that is, the group is parametrized by the *function* $\alpha(\vec{x})$. Consequently, this group is infinite dimensional (not just the representation, but the *group* itself)!

What is the topology of this group? The topology of U(1)×U(1) is the product space of two circles, $S^1 \times S^1$, which is a torus, or more formally, a 2-torus. Thus, the full group $\mathrm{U}(1)^{\infty}$ must have the topology of an infinite-dimensional hypertorus! The corresponding Lie algebra is an infinite-dimensional vector space with functions, $c(\vec{x})$, as its elements. A possible basis is given by the Dirac pulses $\delta^3(\vec{x} - \vec{\xi})$, where $\vec{\xi}$ ranges over all possible events. Note that all basis generators commute with each other.

Local U(1) gauge transformations are *not* a symmetry of the Dirac Lagrangian density. If we transform our toy Dirac Lagrangian density, it does *not* stay the same: $\mathcal{L}' = ie^{in\alpha(t,x)}\psi^* \, \partial(e^{-in\alpha(t,x)}\psi)/\partial t - ie^{in\alpha(t,x)}\psi^* \, \partial(e^{-in\alpha(t,x)}\psi)/\partial x - me^{in\alpha(t,x)}\psi^* e^{-in\alpha(t,x)}\psi = i\psi^* \, \partial\psi/\partial t + n\partial\alpha/\partial t \, \psi^*\psi - i\psi^* \, \partial\psi/\partial x - n\partial\alpha/\partial x \, \psi^*\psi - m\psi^*\psi \neq \mathcal{L}$. There are two extra terms (shown in green), which vanish only if $\partial\alpha/\partial t = \partial\alpha/\partial x = 0$, that is, when $\alpha(t, x)$ is constant, taking us back to *global* gauge transformations. For local gauge transformations to make sense, they must simultaneously act on the electron field *and* the connection field. In the next example, we will discuss the connection field and how it transforms.