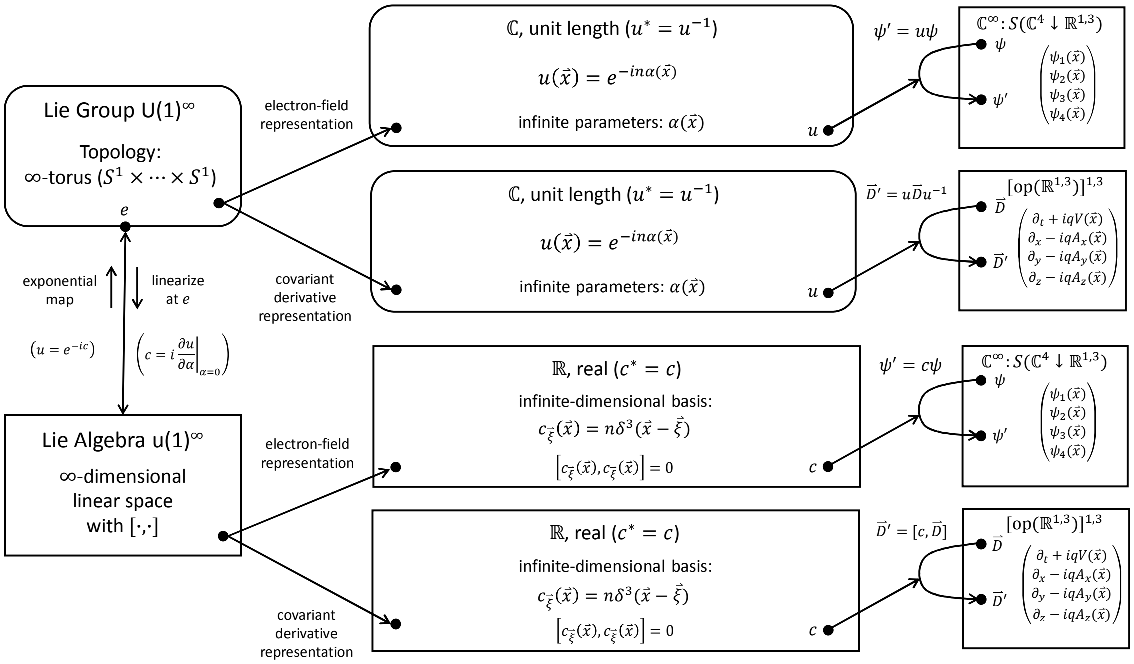
6.9 U(1)$^{\infty}$: Application to the Electromagnetic Potential; Covariant Derivative



To mathematically describe an electron field with a path-dependent phase we introduced two ideas: local "electron-phase frames" and a connection field. We discussed the local frames in the previous example; now, we turn to the connection field.

The connection field tells us how to "parallel transport" the electron field value ψ at event $\vec{x} = (t, x, y, z)^T$ to the neighboring event $\vec{x} + d\vec{x}$. We write $\psi'(\vec{x} + d\vec{x}) = \psi(\vec{x}) - [\vec{\Gamma}(\vec{x}) \cdot d\vec{x}]\, \psi(\vec{x})$, where $\psi'(\vec{x} + d\vec{x})$ is the "parallel-transported" field value, that is, the reference against which we can compare $\psi(\vec{x} + d\vec{x})$, and $\vec{\Gamma}(\vec{x})$ is the (4-vector) connection field. A linear transport formula is sufficient because the neighboring points are infinitesimally close. The minus sign in the formula is just a convention. $\vec{\Gamma}$ must be purely imaginary such that only the phase of ψ is affected (while the magnitude remains fixed). When moving orthogonal to $\vec{\Gamma}$, the local "electron-phase frames" do not change ($\vec{\Gamma} \cdot d\vec{x} = 0$). In contrast, when moving parallel to $\vec{\Gamma}$, they change the most. These frame changes must be taken into account when calculating the gradient or rate of change of the field. This means that we must upgrade the concept of an ordinary derivative to that of a *covariant derivative*.

The ordinary derivative $\vec{\nabla} = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)^T = (\partial_t, \partial_x, \partial_y, \partial_z)^T$ tells us that the value of a (scalar) field changes like $\psi(\vec{x} + d\vec{x}) = \psi(\vec{x}) + \vec{\nabla}\psi(\vec{x}) \cdot d\vec{x}$ when taking the infinitesimal step $d\vec{x}$. The covariant derivative $\vec{D} = (D_t, D_x, D_y, D_z)^T$ tells us how the field changes *relative to the "parallel-transported" value* ψ' when taking the same step, namely like $\psi(\vec{x} + d\vec{x}) = \psi'(\vec{x} + d\vec{x}) + \vec{D}\psi(\vec{x}) \cdot d\vec{x}$. Plugging in $\psi'(\vec{x} + d\vec{x})$ from above, we get $\psi(\vec{x} + d\vec{x}) = \psi(\vec{x}) - [\vec{\Gamma}(\vec{x}) \cdot d\vec{x}]\, \psi(\vec{x}) + \vec{D}\psi(\vec{x}) \cdot d\vec{x}$. Comparing the two derivatives, reveals that the covariant derivative can be written in terms of the ordinary derivative and the connection field as $\vec{D}(\vec{x})\psi(\vec{x}) = \vec{\nabla}\psi(\vec{x}) + \vec{\Gamma}(\vec{x})\psi(\vec{x})$ or, rewritten as an operator equation, as $\vec{D}(\vec{x}) = \vec{\nabla} + \vec{\Gamma}(\vec{x})$. The $\vec{\Gamma}$ part of $\vec{D}$ compensates for the variation of the local "electron-phase frame" from event to event!

How does the covariant derivative $\vec{D}(\vec{x})$ transform under $\mathrm{U}(1)^\infty$? The upper branch of the diagram shows again how the electron field $\psi(\vec{x})$ transforms. To be consistent, the covariant derivative must transform the same way as the object it acts on: $\vec{D}'(\vec{x})u(\vec{x})\psi(\vec{x}) = u(\vec{x})\vec{D}(\vec{x})\psi(\vec{x})$. Thus, $\vec{D}'(\vec{x}) = u(\vec{x})\vec{D}(\vec{x})u^{-1}(\vec{x})$, which we recognize as the transformation law of the adjoint representation (see the lower branch of the diagram). It is important to realize that this expression does *not* reduce to the identity transformation, $u\vec{D}u^{-1} \neq \vec{D}$, because the operator $\vec{D}$ does not commute with u^{-1}.

Knowing how the covariant derivative transforms, we can now figure out how the connection field transforms. The covariant derivative $\vec{D}(\vec{x}) = \vec{\nabla} + \vec{\Gamma}(\vec{x})$ acting on the electron field $\psi(\vec{x})$ transforms like $e^{-in\alpha(\vec{x})}[\vec{\nabla} + \vec{\Gamma}(\vec{x})]e^{in\alpha(\vec{x})}\psi(\vec{x})$. This expands to $e^{-in\alpha(\vec{x})}\vec{\nabla}e^{in\alpha(\vec{x})}\psi(\vec{x}) + \vec{\Gamma}(\vec{x})\psi(\vec{x})$ and after applying the product rule to $\vec{\nabla}\left[e^{in\alpha(\vec{x})} \cdot \psi(\vec{x})\right]$ becomes $in\vec{\nabla}\alpha(\vec{x})\psi(\vec{x}) + \vec{\nabla}\psi(\vec{x}) + \vec{\Gamma}(\vec{x})\psi(\vec{x})$. Comparing this to $[\vec{\nabla} + \vec{\Gamma}'(\vec{x})]\psi(\vec{x})$, we find $\vec{\Gamma}'(\vec{x})\psi(\vec{x}) = in\vec{\nabla}\alpha(\vec{x})\psi(\vec{x}) + \vec{\Gamma}(\vec{x})\psi(\vec{x})$ and thus the connection field transforms like $\vec{\Gamma}'(\vec{x}) = \vec{\Gamma}(\vec{x}) + in\vec{\nabla}\alpha(\vec{x})$. Note that $\vec{\Gamma}(\vec{x})$ and $\vec{\nabla}$ do *not* furnish representations of $\mathrm{U}(1)^\infty$, but their sum $\vec{D}(\vec{x}) = \vec{\nabla} + \vec{\Gamma}(\vec{x})$ does.

Now, we make the bold claim that the connection field is related to the *electromagnetic vector potential*, $\vec{A} = (V, -\vec{A})^T = (V, -A_{x,} -A_{y,} -A_{z,})^T$, which after quantization becomes the *photon field*. Specifically, we claim that $\vec{A} = \vec{\Gamma}/(iq)$, where $q = ne$ is the charge of the spinor field, e is the elementary charge, and we used units in which the speed of light is one. Does this make sense? Rewriting the above transformation $\vec{\Gamma}'(\vec{x}) = \vec{\Gamma}(\vec{x}) + in\vec{\nabla}\alpha(\vec{x})$ in terms of the vector potential yields $\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \vec{\nabla}\alpha(\vec{x})/e$, which is just the gauge transformation of the electromagnetic potential well known from classical electrodynamics [TM, Vol. 3; QFTGA, Ch. 14.1]! The covariant derivative can now be written as $\vec{D}(\vec{x}) = \vec{\nabla} + iq\vec{A}(\vec{x})$, as shown in the lower branch of the diagram. When we promote a connection field like $\vec{\Gamma}(\vec{x})$ to a physical field like $\vec{A}(\vec{x})$, it is called a *gauge field*.

Starting with the trivial connection, $\vec{\Gamma}(\vec{x}) = \vec{A}(\vec{x}) = 0$ for all $\vec{x}$, we can arbitrarily "rotate" the local "phase frames" by multiplying $\psi(\vec{x})$ with $e^{-in\alpha(\vec{x})}$ and simultaneously update the potential to $\vec{A}(\vec{x}) = \vec{\nabla}\alpha(\vec{x})/e$, where $\alpha(\vec{x})$ is any smooth real function. The resulting infinitely many different mathematical expressions all describe the same electromagnetic situation (an electron field without path dependence and no electric or magnetic field) and thus are equivalent! The potential $\vec{A}(\vec{x}) = \vec{\nabla}\alpha(\vec{x})/e$, which has no physical effect, is known as *pure gauge*.

The analogous situation in geometry is flat space. For Cartesian coordinates, the connection field is zero everywhere, that is, no corrections are needed when parallel transporting a vector. We can then arbitrarily (and smoothly) deform the coordinates into curved ones and simultaneously update the connection field to reflect the twisting and turning of the new coordinates. Again, the resulting infinitely many mathematical expressions all describe the same object (flat space) and thus are equivalent! (See the Appendix "Christoffel Symbols for Polar Coordinates" for an example.) For a financial analog likening connections to money exchange rates, see https://arxiv.org/abs/1410.6753 and [PfF].

Physical reality cannot depend on our choice of coordinates and therefore the laws of electrodynamics must not be affected by the joint transformation $\psi'(\vec{x}) = \psi(\vec{x})e^{-in\alpha(\vec{x})}$ and $\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \vec{\nabla}\alpha(\vec{x})/e$. This is the *gauge symmetry* of quantum electrodynamics.