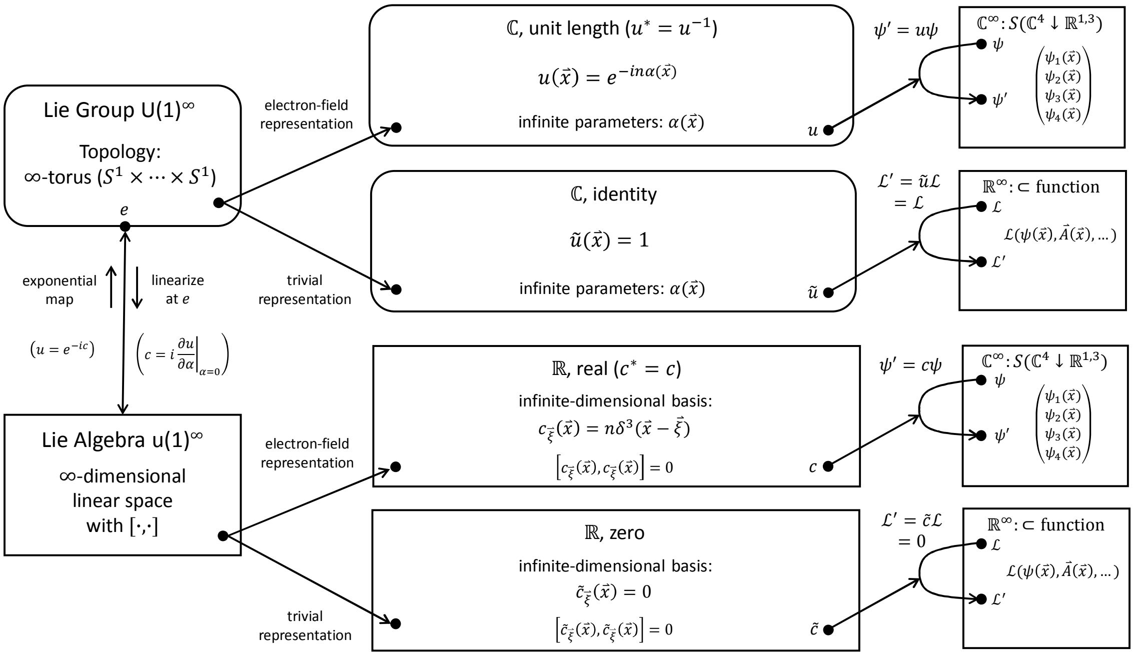
6.11 U(1)$^\infty$: Application to Electromagnetic Interactions; Lagrangian of QED



We said earlier that the local gauge transformation $\psi'(\vec{x}) = \psi(\vec{x})e^{-in\alpha(\vec{x})}$ and $\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \vec{\nabla}\alpha(\vec{x})/e$ must not affect the laws of quantum electrodynamics (QED), because physical reality cannot depend on our choice of coordinates. More precisely, the Lagrangian density (and thus the action) of QED must remain *invariant* under local gauge transformations, that is, it must transform under the trivial representation of $\mathrm{U}(1)^\infty$ (see the lower branch of the diagram). Now we want to use this gauge symmetry as a constraint for guessing the correct Lagrangian density of QED.

The Dirac Lagrangian density, which describes the dynamics of the free electron field, does *not* have the required local gauge invariance. We demonstrated this earlier for our toy Dirac Lagrangian density, $\mathcal{L}_\psi = i\psi^*(\partial_t - \partial_x)\psi - m\psi^*\psi$. Applying a local gauge transformation to it yields $\mathcal{L}'_\psi = ie^{in\alpha(t,x)}\psi^*(\partial_t - \partial_x)e^{-in\alpha(t,x)}\psi - me^{in\alpha(t,x)}\psi^*e^{-in\alpha(t,x)}\psi = i\psi^*(\partial_t - \partial_x)\psi + n\psi^*[\partial_t\alpha(t,x) - \partial_x\alpha(t,x)]\psi - m\psi^*\psi \neq \mathcal{L}_\psi$. The mass term, $m\psi^*\psi$, remains invariant, but the kinetic-plus-gradient energy term, $i\psi^*(\partial_t - \partial_x)\psi$, produces the additional term in green that spoils the local gauge symmetry. The same is true for the full Dirac Lagrangian density.

The easiest way to make this Lagrangian density gauge invariant is by replacing the ordinary derivatives with *covariant derivatives*. We know that (by definition) the covariant derivative transforms the same way as the object it acts on. Thus, the first term of our toy Dirac Lagrangian density, which now reads $i\psi^*(D_t - D_x)\psi$, transforms like $i\psi^*\psi$, which is invariant, just like the mass term $m\psi^*\psi$ is.

Let's spell this out in more detail (again using our toy Dirac Lagrangian density): We start with $\mathcal{L}_\psi = i\psi^*(\partial_t - \partial_x)\psi - m\psi^*\psi$, upgrade it with covariant derivatives to $\mathcal{L}_{\psi I} = i\psi^*(D_t - D_x)\psi - m\psi^*\psi$, and expand $D_t\psi = \partial_t\psi + iqV\psi$ and $D_x\psi = \partial_x\psi - iqA_x\psi$, which yields

$$\mathcal{L}_{\psi I} = i\psi^*(\partial_t - \partial_x)\psi - m\psi^*\psi - q\psi^*(V + A_x)\psi.$$

This is just the original Dirac Lagrangian density plus a new term at the end that brings the electromagnetic potential into the mix. Now, let's check again if local U(1) gauge symmetry holds. Remembering to transform $\psi(\vec{x})$ and $\vec{A}(\vec{x})$ simultaneously, we have

$$\mathcal{L}'_{\psi I} = i e^{in\alpha}\psi^*(\partial_t - \partial_x)e^{-in\alpha}\psi - m e^{in\alpha}\psi^* e^{-in\alpha}\psi - q e^{in\alpha}\psi^*\left(V + \frac{\partial_t\alpha}{e} + A_x - \frac{\partial_x\alpha}{e}\right)e^{-in\alpha}\psi,$$

which simplifies to (remember $q = ne$)

$$\mathcal{L}'_{\psi I} = i\psi^*(\partial_t - \partial_x)\psi + n\psi^*(\partial_t\alpha - \partial_x\alpha)\psi - m\psi^*\psi - q\psi^*(V + A_x)\psi - n\psi^*(\partial_t\alpha - \partial_x\alpha)\psi = \mathcal{L}_{\psi I}.$$

Happily, the new terms in orange exactly cancel the offending terms in green, confirming that the revised Lagrangian is invariant! All of this works equally well for the full Dirac Lagrangian density [PfS, Ch. 7.1.3].

The above procedure to make the Lagrangian density gauge invariant added the new term $-q\psi^*(V + A_x)\psi$. This term describes the *interaction* between the electron field ψ and the electromagnetic potential field $(V, -A_x)$, and the interaction strength is given by the charge q of the ψ field. This interaction term is the simplest one that satisfies the required local U(1) gauge symmetry and hence is referred to as the *minimal coupling* term. Luckily, it is also the term Nature chose!

How can we visualize this interaction? Imagine an electron field in the form of a plane wave with a fixed wavelength. When quantized, such a field yields free electrons with a fixed momentum. In the absence of an EM field (and picking the trivial gauge), all the "electron-phase frames" are the same everywhere, but when we turn on an EM field, they change from event to event. As a result, the local phase of the electron plane wave, which is measured against these frames, varies too. These local phase shifts affect the wavelength (and the wave vector) of the plane wave. Translated into the particle picture, the momentum of the electrons changes, that is, it looks as if a force acted on them!

To obtain the full theory of QED we need to combine three pieces: the Lagrangian density for the free electron field, $\mathcal{L}_\psi$, the interaction term that we discussed above, $\mathcal{L}_I$, and the Lagrangian density for the free electromagnetic potential, $\mathcal{L}_A$. In toy form, this can be written as

$$\mathcal{L} = i\psi^*(\partial_t - \partial_x)\psi - m\psi^*\psi - q\psi^*(V + A_x)\psi + \tfrac{1}{2}(\partial_t A_x + \partial_x V)^2.$$

In its full form, the Lagrangian density of QED reads [QFTGA, Ch. 38.4; PfS, Ch. 7.1.3]

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu).$$

Symmetries helped us to find all three parts: the first and third part is constrained by the global space-time symmetry Spin$^+$(1,3) = SL(2,ℂ) and the second part is constrained by the local U(1) gauge symmetry, as discussed above. (The latter symmetry also prevents the third part from having a photon-mass term, which would be allowed by the space-time symmetry alone.)

The equations of motion derived from the above Lagrangian density are the *Maxwell equations*, which describe how charges and currents produce electromagnetic fields, and the *Lorentz force law*, which describes how charges and currents are pushed around by electromagnetic fields [PfS, Ch. 7.1.4 & 6]. See the Appendix "From Lorentz and Gauge Symmetry to Maxwell's Equations" for the equations of motion.