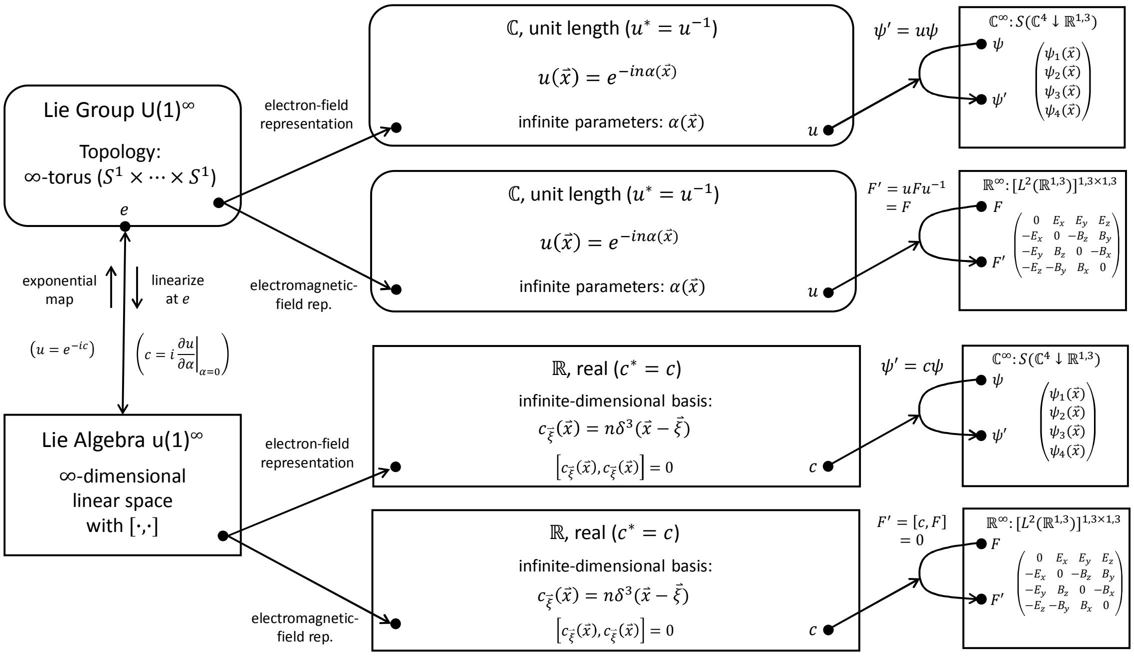
6.10 U(1)$^\infty$: Application to the Electromagnetic Field Strength; Curvature



In the following, we'll calculate the difference between two electron-field values, one at event a and one at b, following two different paths connecting the events. To do that we use the covariant derivative introduced in the previous example. Then, we determine the mismatch between these two differences and normalize it to the local field value and the area enclosed by the two paths to obtain the *curvature* of the electron-field space. This curvature is closely related to the electromagnetic field!

Let's call the starting event for both paths $\vec{x}_a$. The first path leads a small distance ε in the x direction, makes a 90° left turn and continues the same distance ε in the y direction, whereas the second path goes first in the y direction and then in the x direction. Both paths end up at the same event $\vec{x}_b$. Along the first leg of the first path, the field changes by $D_x\psi(\vec{x}_a)\varepsilon$, where D_x is the covariant derivative in the x direction. Along the second leg of the first path, the field changes by $D_y\psi(\vec{x}_a + \vec{\varepsilon}_x)\varepsilon$, where $\vec{x}_a + \vec{\varepsilon}_x$ is the end point of the first leg. Referring this change back to the starting event $\vec{x}_a$, we get $D_y[\psi(\vec{x}_a) + D_x\psi(\vec{x}_a)\varepsilon]\varepsilon$. Thus, the total difference along the first path is $D_x\psi(\vec{x}_a)\varepsilon + D_y[\psi(\vec{x}_a) + D_x\psi(\vec{x}_a)\varepsilon]\varepsilon = D_x\psi(\vec{x}_a)\varepsilon + D_y\psi(\vec{x}_a)\varepsilon + D_yD_x\psi(\vec{x}_a)\varepsilon^2$. Now, doing the same calculation for the second path yields $D_y\psi(\vec{x}_a)\varepsilon + D_x\psi(\vec{x}_a)\varepsilon + D_xD_y\psi(\vec{x}_a)\varepsilon^2$. Subtracting the first difference from the second one, the terms linear in ε cancel out and we are left with the mismatch $[D_xD_y\psi - D_yD_x\psi]\varepsilon^2 = [D_x, D_y]\psi\varepsilon^2$. Dividing this by the enclosed area ε^2, yields $R_{xy}\psi := [D_x, D_y]\psi$, where we introduced R_{xy}, the xy-component of the *curvature tensor*. Expanding the covariant derivatives in terms of the connection field, we get $R_{xy}\psi = (\partial/\partial x + \Gamma_x)(\partial/\partial y + \Gamma_y)\psi - (\partial/\partial y + \Gamma_y)(\partial/\partial x + \Gamma_x)\psi$. Multiplying out, applying the product rule, and simplifying yields $R_{xy} = \partial\Gamma_y/\partial x - \partial\Gamma_x/\partial y$. Note that the electron field, $\psi(\vec{x})$, dropped out. Thus, the curvature depends only on the connection field $\vec{\Gamma}(\vec{x})$, which characterizes the space in which the electron field lives!

Next, we generalize the above analysis for an arbitrary pair of space-time indices to get the full 4×4 curvature tensor $R_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu$, where $\mu, \nu \in \{t, x, y, z\}$ and $\partial_\mu := \partial/\partial\mu$. Because this tensor is antisymmetric, $R_{\mu\nu} = -R_{\nu\mu}$, only six of the sixteen tensor components are independent. Geometrically, this comes about because there are only six distinct orthogonal planes in 4D space (xy, yz, zx, xt, yt, and zt) and because reversing the orientation of the path only changes the sign of the mismatch. Note that $R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$ is just the *exterior derivative* of Γ_μ and can be written more compactly as $R = \boldsymbol{d}\vec{\Gamma}$, where $\boldsymbol{d}$ denotes the exterior derivative. (Yet another way to write the curvature tensor is $R = \vec{\nabla} \otimes \vec{\Gamma} - [\vec{\nabla} \otimes \vec{\Gamma}]^T$, where $\otimes$ denotes the tensor product.) Moreover, if we regard $\vec{\Gamma}(\vec{x})$ as a one-form (= covector field), then $R(\vec{x})$ is a two-form. See the Appendix "The Exterior Derivative in 3D Euclidean Space; Gradient, Curl, and Divergence".

How does the curvature tensor change under a local U(1) gauge transformation? The curvature tensor can be expressed in terms of the covariant derivate as $R_{\mu\nu} = D_\mu D_\nu - D_\nu D_\mu$ and the covariant derivative transforms under the adjoint representation like $D'_\mu = u D_\mu u^{-1}$. Thus, we have $R'_{\mu\nu} = u D_\mu u^{-1} u D_\nu u^{-1} - u D_\nu u^{-1} u D_\mu u^{-1} = u(D_\mu D_\nu - D_\nu D_\mu)u^{-1} = u R_{\mu\nu} u^{-1}$. Plugging in u, we find: $R'_{\mu\nu} = u R_{\mu\nu} u^{-1} = e^{-in\alpha(\vec{x})} R_{\mu\nu} e^{in\alpha(\vec{x})} = R_{\mu\nu}$, that is, the curvature tensor remains *invariant*.

To get a better understanding of what $R_{\mu\nu}$ represents, let's consider the infinitesimal *loop* from $\vec{x}_a$ to $\vec{x}_b$ along the first path and then back from $\vec{x}_b$ to $\vec{x}_a$ along the second path in reverse. "Parallel transporting" the electron field $\psi(\vec{x}_a)$ around such a loop with area ε^2 located in the $\mu\nu$-plane results in $\psi'(\vec{x}_a) = \psi(\vec{x}_a) + R_{\mu\nu}(\vec{x}_a)\psi(\vec{x}_a)\varepsilon^2$, which follows from the above mismatch calculation for the two alternative paths. Since Γ_μ is purely imaginary, so is $R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$ and the transported field value differs from the original one in phase only. In conclusion, $R_{\mu\nu}$ measures by how much the "electron-phase frame" rotates when moving around a small loop in the $\mu\nu$ plane!

Why is $R_{\mu\nu}$ called the curvature tensor? Let's return to 2D geometry. If we parallel transport the tangent vector $\vec{v}_a(\vec{x}_a)$ around a small loop and end up with $\vec{v}'_a(\vec{x}_a)$ being rotated against $\vec{v}_a(\vec{x}_a)$, then the surface near the point $\vec{x}_a$ is curved. The difference $\vec{v}'_a(\vec{x}_a) - \vec{v}_a(\vec{x}_a)$, called *holonomy*, is proportional to the length $|\vec{v}_a|$ and the enclosed area ε^2 and after normalization represents the *curvature* of the 2D surface! For a financial analog in which the opportunity for *currency arbitrage* is likened to curvature of "money space", see https://arxiv.org/abs/1410.6753 and [PfF].

Now, we make the bold claim that the curvature tensor $R_{\mu\nu}$ is related to the *electromagnetic field-strength tensor* $F_{\mu\nu}$ (a.k.a. *Faraday tensor*)! Specifically, we claim that $F_{\mu\nu} = R_{\mu\nu}/(iq)$, where q is the charge of the spinor field and we used units in which the speed of light is one. Does this make sense? Rewriting the relationship between curvature and connection, $R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$, in terms of $F_{\mu\nu}$ and A_μ (= $\Gamma_\mu/(iq)$) yields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which is just the relationship between field strength and potential well known from classical electrodynamics [TM, Vol. 3]! This relationship can be written more compactly as $F = \boldsymbol{d}\vec{A}$, where $\boldsymbol{d}$ is the exterior derivative. Note that the invariance of $F_{\mu\nu} = R_{\mu\nu}/(iq)$ under $\mathrm{U}(1)^\infty$ that we discussed above agrees with the fact that the electric and magnetic fields do *not* depend on our choice of gauge. See the lower branch of the diagram.

In conclusion, the magnetic field can be interpreted as the rotation of the "electron-phase frame" when moving around an infinitesimal loop in a space-space plane. For example, the magnetic field in the z direction is $B_z = F_{yx} = R_{yx}/(iq)$. Similarly, the electric field can be interpreted as the rotation of the "electron-phase frame" when moving around a loop in a space-time plane. For example, the electric field in the x direction is $E_x = F_{tx} = R_{tx}/(iq)$.