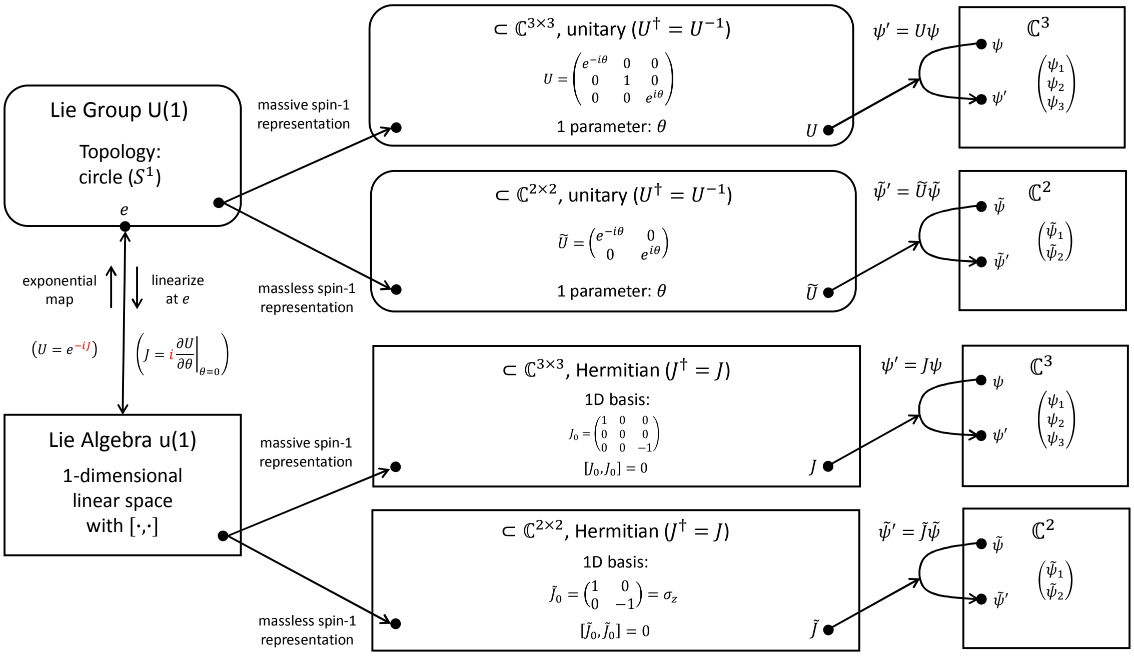
6.4 U(1): Application to Photon Spin and Polarization



With this example, we return to quantum mechanics and thus include a factor i with the generators: $J = iX$. As we know, this makes J Hermitian and its real eigenvalues can be interpreted as measurement outcomes (e.g., spin values). To remain consistent, the exponential map is modified to $U = e^{-iJ}$.

As we know, SU(2) representations can be parametrized by the rotation angles θ_x, θ_y, and θ_z about the x, y and z axes, respectively. Now, we are going to fix the first two angles, $\theta_x = \theta_y = 0$, and rename θ_z to θ. Applying this restriction to the 3-dimensional (spin-1) representation of SU(2), discussed in an earlier example, yields the 3-dimensional representation of U(1) shown in the upper branch of the diagram. Inspecting the transformation matrix, we see that it is the direct sum of three 1-dimensional irreducible representations, namely the complex-conjugate representation ($k = -1$), the trivial representation ($k = 0$), and the defining representation ($k = 1$). This makes clear that U(1) is a *subgroup* of SU(2). Clearly, representations of U(1) can be used to rotate quantum states, ψ, about a *fixed* axis in 3D space. In our example, the matrix $U(\theta)$ rotates the state of a massive spin-1 particle about the z axis.

The Lie algebra u(1) of our 3-dimensional representation has a single basis generator, J_0, which equals J_z of the corresponding su(2) algebra. A general element of this (1-dimensional) Lie algebra is just a scaled version of the basis generator $J = J_0\theta$. The eigenvalues of the basis-generator matrix are just its diagonal components, $+1, 0, -1$, which represent the possible spin values that can be measured along the (fixed) z axis. We have seen in an earlier example that the integer eigenvalues of J are due to the periodicity of rotation; therefore, the quantization of spin has the same origin!

A moving particle breaks the isotropy of 3D space. Rather than the full 3D rotational symmetry, we now have only a 2D rotational symmetry about the particle's direction of motion. However, if the particle is massive and thus moves slower than the speed of light, we can "catch up" with it and put it into a rest frame. In this frame there is no preferred direction in space and the full 3D symmetry is restored! In contrast, if the particle is *massless* and thus moves at the speed of light, we cannot do that, and the 3D rotational symmetry breaks down to 2D for good. In other words, SU(2) rotational symmetry breaks down to U(1). From the "point of view" of the particle moving at the speed of light all distances in its direction of motion Lorentz contract to zero, making this direction of space rather meaningless. Thus, massless particles inhabit a kind of 2-dimensional world with U(1) symmetry!

How can we describe the quantum state of a massless spin-1 particle, such as a photon? We know, that a massive spin-1 particle has three basis states, but the photon, living in its "2D world", can only spin in the plane orthogonal to its direction of motion, that is, its spin is either along the direction of motion, $\psi = (1, 0)^T$, or in the opposite direction, $\psi = (0, 1)^T$, reducing the number of basis states from three to two [FLP, Vol. III, Ch. 17-4]. Spin measured along the direction of motion (= projected on the momentum vector) is known as *helicity*. So, a photon with spin parallel to the direction of motion has helicity +1, whereas a photon with spin antiparallel to the direction of motion has helicity –1. The states of helicity are also called states of *circular polarization*: right-handed circular (RHC) polarization for helicity +1 and left-handed circular (LHC) polarization for helicity –1. It turns out that under rotation (about the direction of motion) the helicity states transform like the states of a massive spin-1 particle, but without the middle component that corresponds to the suppressed basis state [FLP, Vol. III, Ch. 11-4 & 17-6]. The representation of U(1) acting on spin-1 helicity states is shown in the lower branch of the diagram. Note that this is the same representation that we discussed in the previous example.

The operator for the helicity (or circular-polarization) observable is the generator of the U(1) rotational symmetry, which is equal to the Pauli matrix σ_z (see the lower branch of the diagram). The RHC state $\psi = (1, 0)^T$ is an eigenvector with eigenvalue +1 of this operator and thus has the definite helicity +1. Rotating the RHC state by the angle θ about the direction of motion yields $\psi' = (e^{-i\theta}, 0)^T$, which is again an eigenstate with helicity +1 (regardless of the rotation angle). Similarly, the LHC state $\psi = (0, 1)^T$ has the definite helicity –1 and also remains unaffected by rotation.

The operator for the *linear x-y-polarization* observable turns out to be the Pauli matrix σ_x ($+1$ for x or horizontal polarization and -1 for y or vertical polarization) [?]. Note that this is *not* a generator of rotational symmetry. The state $\psi = 1/\sqrt{2} \cdot (1, 1)^T$, an equal superposition of the two helicity basis states, is an eigenstate with eigenvalue +1 of this operator and thus is a state of definite horizontal polarization. Rotating this state by 90° results in $\psi' = 1/\sqrt{2} \cdot (e^{-i\pi/2}, e^{i\pi/2})^T = i/\sqrt{2} \cdot (-1, 1)^T$, which is a state of definite vertical polarization (eigenvalue $= -1$). Similarly, σ_y is the operator for the *linear diagonal-polarization* observable ($+1$ for $x = y$ or diagonal polarization and -1 for $x = -y$ or anti-diagonal polarization). In optics, the state vector ψ is usually given in the x-y basis and referred to as *Jones vector* [Wikipedia: *Jones calculus*].

It is instructive to compare massless spin-1 particles (e.g., photons) with massive spin-½ particles (e.g., electrons), both of which are two-state systems with an SU(2) symmetry under a basis change. The (unnormalized) generators of this symmetry are the Pauli matrices σ_x, σ_y, and σ_z. In the case of spin-½ particles these generators also describe rotations about the x, y, and z axis and thus are spin observables. Any spin state can be related to a particular direction in physical 3D space along which the spin is guaranteed to be up. In the case of massless spin-1 particles only σ_z is a spin (or rather helicity) observable, the other two generators correspond to linear polarizations and are not related to spatial rotation. A general polarization state (circular, elliptic, and linear) can be described by a so-called *Stokes vector* pointing in an *abstract* 3D space [RtR, Ch. 22.9].