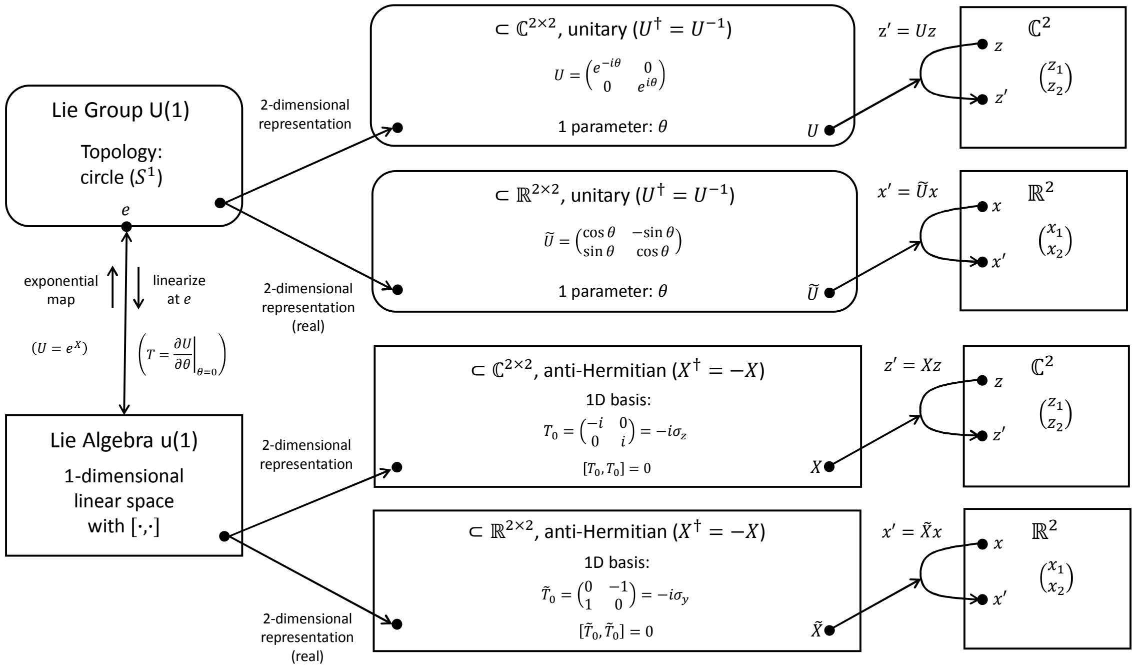
6.3 U(1): Representation on Real 2-Dimensional Vectors



We know that all irreducible representations of U(1) are one dimensional. By taking the direct sum of *n* such representations, we can construct a *reducible* n-dimensional representation. Let's take the direct sum of the $k = -1$ and $k = +1$ representations to construct a 2-dimensional representation. This representation, which acts on a 2-dimensional complex vector like $z' = Uz$, is shown in the upper branch of the diagram. We'll discuss an application of this representation (rotation of photon states) in the next example.

While this representation certainly looks complex, it might secretly be *pseudoreal*. Since we now have two dimensions to play with, a similarity transformation that maps it to its complex conjugate, $SUS^{-1} = U^*$, might exist. Sure enough, the following transformation does the job:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}^*.$$

Note that the matrix S is its own inverse and the transformation just swaps the two matrix components on the diagonal. Unsurprisingly, the same transformation also maps the basis generator T_0 to its complex conjugate, $ST_0S^{-1} = T_0^*$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}^*.$$

We thus have established that this representation is not really complex, but pseudoreal. But could it even be *real*? To figure that out, we have to find a similarity transformation, SUS^{-1}, that makes it manifestly real. This transformation is harder to find, but it does exist [GTNut, p. 196]:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Applying the same transformation to the basis generator, ST_0S^{-1}, yields:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is again totally real. This real 2-dimensional representation of U(1) is shown in the lower branch of the diagram.

What does this real 2×2 transformation matrix do? It rotates real 2D vectors by the angle θ about the origin! This is reminiscent of the 3-dimensional (irreducible) representation of SU(2), which despite looking complex could be made real and shown to rotate 3D vectors about an axis passing through the origin.

Generalizing this example, we can construct an n-dimensional representation of U(1) by combining n irreducible representations. To do that, it is convenient to start with a diagonal $n \times n$ matrix $J_0 = \mathrm{diag}(j_1, j_2, \dots, j_n)$, where the diagonal components, j_i, are integers (multiple occurrences of the same integer are allowed). From that, we construct the transformation matrix $U = e^{-iJ_0\theta}$, where the minus sign is just for consistency with later examples.

Finally, the most general finite-dimensional representation of U(1) can be written as $U = e^{-iJ_0\theta}$, where J_0 is now an $n \times n$ matrix with integers *eigenvalues* $j_1, j_2, \dots, j_n$ [QTGR, Ch. 2.5]. This general form is related to the above diagonal form by a similarity transformation.