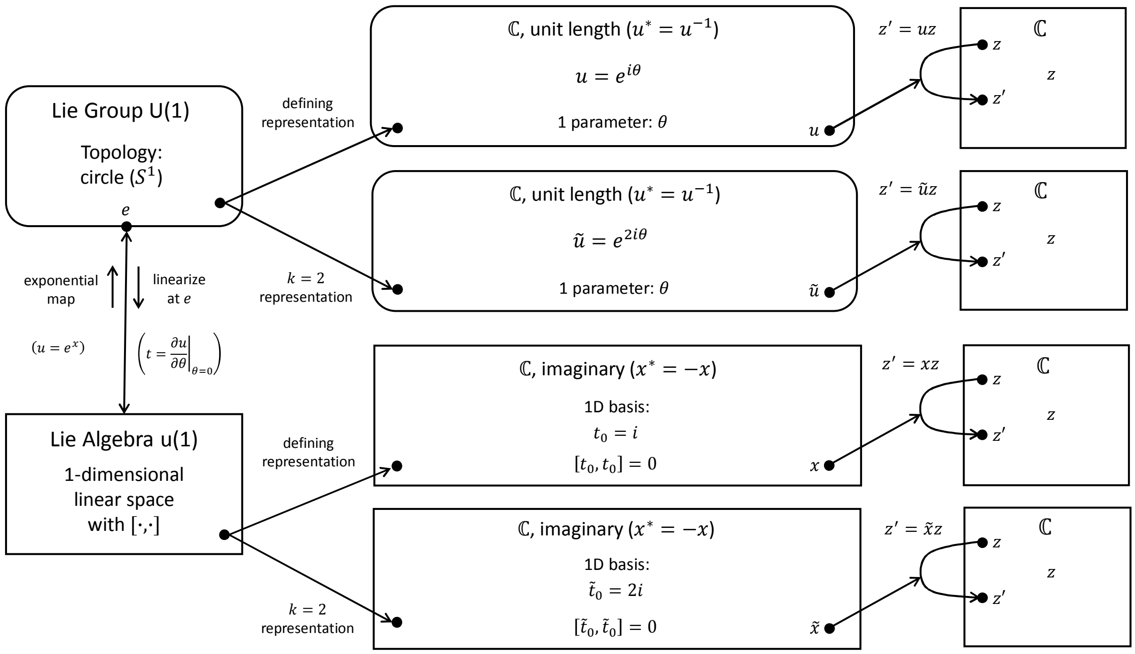
6. Rotation in Two Dimensions and Electromagnetism

6.1 U(1): The Group of Complex Numbers of Unit Length



We now turn to the important *unitary group of degree one*, U(1), also known as the *circle group*. We have already encountered this group when we discussed how the unitary transformations U(2) can be decomposed into transformations with determinant one, SU(2), and a phase factor, U(1). Moreover, U(1) is a subgroup of SU(2); in fact, SU(2) has infinitely many U(1) subgroups. But these subgroups are not *normal* subgroups that can be factored out. In physics, U(1) plays a major role as the *gauge group of electromagnetism*. The U(1) gauge symmetry determines how electrically charged matter interacts with the electromagnetic field.

The group U(1), or rather its defining representation, consists of the complex numbers u that preserve the Hermitian inner product, $(up)^*(uq) = p^*q$, for any pair of complex numbers p and q. Rewriting as $p^*u^*uq = p^*q$, we see that this is the case if $u^*u = 1$, which is equivalent to $|u| = 1$, and therefore u is a unit-length complex number: $u = a_0 + ia_1$ with $a_0^2 + a_1^2 = 1$. Note that the constraint on a_0 and a_1 makes this a *one*-parameter set of transformations. A more convenient way of writing the same unit-length complex number is $u = e^{i\theta}$, where the angle θ is the parameter.

U(1) is a rather unusual Lie group and, if encountered as a first example, can be quite confusing! For starters, it is a one-dimensional group, that is, its elements can be parametrized by a single parameter. Therefore, the Lie algebra is also one dimensional, that is, there is only one basis generator. This basis generator necessarily commutes with itself. Thus, the commutation relations and structure constants, so important for Lie algebras in general, are trivial in the case of u(1). A group like U(1) whose elements commute is called an *Abelian* group. In contrast, all groups that we discussed so far were non-Abelian groups. Finally, it turns out that because the elements of U(1) commute, all irreducible representations of U(1) are one dimensional. (This is a consequence of *Schur's lemma* [QTGR, Ch. 2.1]). Clearly, U(1) is not a typical Lie group!

What is the topology of U(1)? From either of the two parametrizations introduced above, we conclude that U(1) has the topology of s circle, S^1, hence the name *circle group*. What is the simplest object (set of points in the representation space) that remains invariant under U(1) transformations? It is a circle in the complex plane. But these two circles must not be confused. The topology of a group and the shape of the objects that remain invariant under the group's action are two different things! For example, the simplest invariant object under the defining representation of SO(3) is the sphere (more precisely, the 2-sphere), but the topology of SO(3) is half of a 3-sphere.

Is the manifold of U(1) simply connected? No, a loop on a circle may wrap around the circle and if it does, it cannot be continuously contracted to a point. Does U(1) have a covering group, like SO(3) is covered by SU(2)? Yes, the 1-dimensional translation group $\mathbb{R}$, which has the topology of an infinitely long line and thus is simply connected, has the same Lie algebra as U(1).

What is the Lie algebra of U(1)? To find the basis generator t_0, we take the derivative of $e^{i\theta}$ with respect to θ and evaluate the result at $\theta = 0$. The result is $t_0 = i$. Thus, the (1-dimensional) Lie algebra u(1) consists of all the imaginary numbers: $x = ci$, where $c \in \mathbb{R}$.

The low-dimensional nature of U(1) makes it easy to visualize the group manifold and its tangent space. The group manifold of U(1) is the unit circle in the complex plane and the identity element is the point $1 + 0i$ on the right-hand side of the circle. The tangent space at the identity is the vertical line given by $1 + ci$. After removing the horizontal offset to the identity element, we are left with the Lie-algebra space ci, in agreement with what we said above. What does the exponential map do? It takes us from $x = i\theta$ to $u = e^{i\theta}$, thus "wrapping" the linear tangent space (= Lie algebra) into a circle (= Lie group). A small group element (= element near the identity) can be written as $u = e^{i\varepsilon}$, where ε is a small angle. This element can be approximated as $u = 1 + i\varepsilon$, a point that is located in the tangent space. The corresponding small Lie-algebra element (= element near the origin) is $x = i\varepsilon$. This element generates a vector field of displacements on the representation space: $z' = i\varepsilon z$. At any point z in the complex plane, the displacement vector z' is the vector from the origin to z reduced to infinitesimal length ($\times\ \varepsilon$) and rotated counterclockwise by 90° ($\times\ i$). The flow lines (or integral curves) of this vector field are the circles given by $z(\theta) = e^{i\theta} z(0)$.

We wrote the defining representation of U(1) as $u = e^{i\theta}$, but we could also write $u = e^{if(\theta)}$, where $f(\theta)$ is an arbitrary smooth and monotonic real function. Changing the parametrization in this way is equivalent to putting different coordinates on the group manifold (= circle). But, in the case of U(1) there is no advantage in doing so and we'll always stay with the original parametrization. A consequence of this convention is that the identity element is parametrized by $\theta = 0, \pm 2\pi, \pm 4\pi$, etc.

Does U(1) have other representations? Yes, it turns out that U(1) has infinitely many irreducible representations, which can be written as $u = e^{ik\theta}$, where $k = 0, \pm 1, \pm 2, \ldots$ [QTGR, Ch. 2.2] The value of k is restricted to integers to ensure that the agreed upon θ values parametrize the identity element. To label these representations we use the value of k. (In contrast to SU(2), dimension is not a useful label because all irreducible representations of U(1) are one dimensional.) The diagram shows the defining ($k = 1$) representation (upper branch) and the $k = 2$ representation (lower branch).