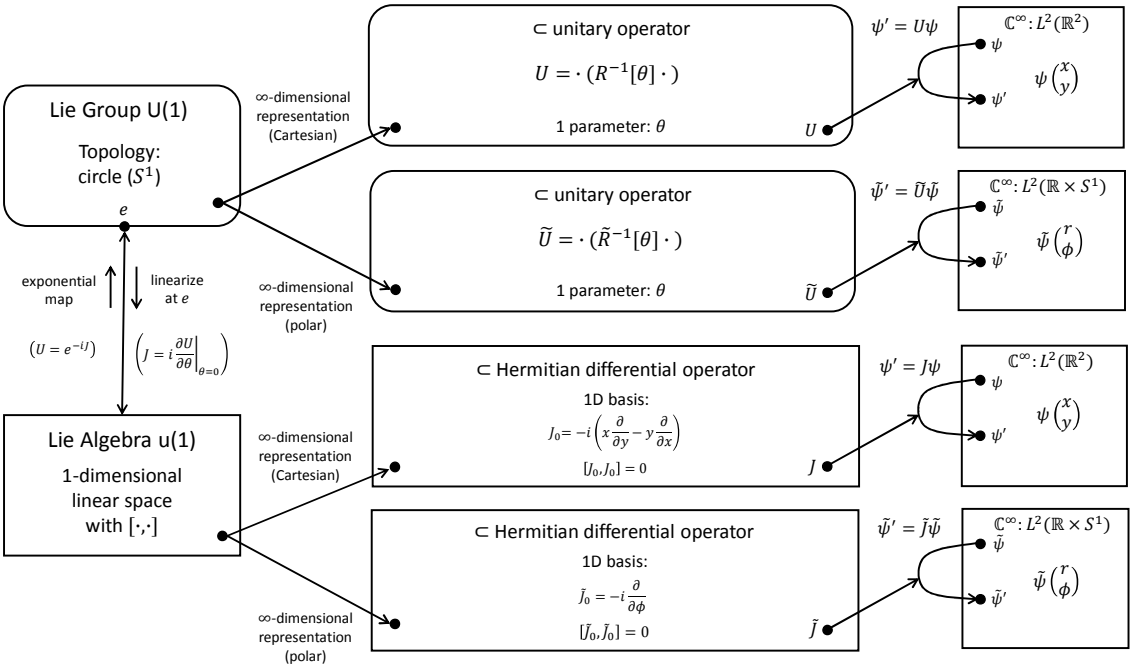
6.5 U(1): Infinite-Dimensional Representations; Circular Harmonics



Let's construct an infinite-dimensional representation of U(1) by starting with the 2-dimensional real representations that we discussed earlier and then upgrading the 2-dimensional representation space to the (infinite-dimensional) space of complex square-integrable functions of two real variables, $\psi(x, y)$. This is analogous to that we did for SU(2) in three dimensions. (Our $\psi(x, y)$ could represent a single-particle wave function at one instant of time in a 2-dimensional world.)

Our infinite-dimensional representation acts on these functions by applying the inverse of the 2-dimensional real representation to the function's argument: $\psi'(\vec{x}) = \psi(R^{-1}[\theta]\,\vec{x})$, where $\vec{x} = (x, y)^T$ and R is the 2D rotation matrix (which we called $\tilde{U}$ in an earlier example). Using that $R^{-1}(\theta) = R(-\theta)$, we find the explicit transformation of the argument as $x' = x\cos(-\theta) - y\sin(-\theta)$ and $y' = x\sin(-\theta) + y\cos(-\theta)$ and thus the function transforms like

$$\psi'(x, y) = \psi(x\cos\theta + y\sin\theta\,, -x\sin\theta + y\cos\theta).$$

To write this as an operator U acting on the function $\psi(x, y)$, that is, $\psi'(\vec{x}) = U\psi(\vec{x})$, we use our informal dot notation: $U = \cdot\,(R^{-1}[\theta]\;\cdot)$, where the first dot stands for the function's name and the second dot for the function's argument. Again, all this is analogous to that we did for the infinite-dimensional representation of SU(2). See the upper branch of the diagram.

To find the Lie-algebra elements of our infinite-dimensional representation, we first differentiate the transformed function with respect to θ: $\partial\psi/\partial x \cdot (-x\sin\theta + y\cos\theta) + \partial\psi/\partial y \cdot (-x\cos\theta - y\sin\theta)$. Then, we evaluate the result at $\theta = 0$ and multiply it by i, which yields $i[\partial\psi/\partial x \cdot y + \partial\psi/\partial y \cdot (-x)]$. Finally, we split off the differential operator from the function to obtain the basis generator

$$J_0 = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

See the upper branch of the diagram.

Next, we switch from Cartesian to polar coordinates, which are more convenient in situations with rotational symmetry. Now, the arguments of the function are the radius and the angle: $\tilde{\psi}(r,\phi) = \tilde{\psi}(\vec{s})$, where $\vec{s} = (r,\phi)^T$, and the function transforms like

$$\tilde{\psi}'(r,\phi) = \tilde{\psi}(r,\phi-\theta).$$

The corresponding operator can be written as $\widetilde{U} = \cdot\,(\,\cdot\,,\ \cdot - \theta)$ or, maybe better, as $\widetilde{U} = \cdot\,(\tilde{R}^{-1}[\theta]\ \cdot\,)$, where $\tilde{R}$ is the operator that adds θ to the ϕ component of the vector. (Note that, in contrast to the Cartesian case, $\tilde{R}$ cannot be written as a 2×2 matrix.) Using the operator $\widetilde{U}$, we can write the transformation as $\tilde{\psi}'(\vec{s}) = \widetilde{U}\tilde{\psi}(\vec{s}) = \tilde{\psi}(\tilde{R}^{-1}[\theta]\ \vec{s})$. The derivative of $\tilde{\psi}'(\vec{s})$ with respect to θ is $-\partial\tilde{\psi}/\partial\phi$. Evaluating this at $\theta = 0$, multiplying by i, and splitting off the differential operator yields the basis generator in polar coordinates:

$$\tilde{J}_0 = -i\frac{\partial}{\partial\phi}.$$

See the lower branch of the diagram.

In quantum mechanics, the (Hermitian) basis generators J_0 or $\tilde{J}_0$ are the operators for the angular-momentum observable. Their eigenfunctions are the wave functions with a *definite* angular momentum, which also provide an orthogonal basis for the representation space. The eigenvalues are the possible measurement outcomes, which also provide convenient labels for the basis functions. What is the explicit form of the eigenfunctions? Working in polar coordinates, the eigenequation is $\tilde{J}_0\Psi_m(\vec{s}) = m\Psi_m(\vec{s})$, where $\tilde{J}_0 = -i\partial/\partial\phi$. Keeping in mind that the function $\tilde{\psi}(\vec{s})$, and thus the eigenfunctions $\Psi_m(\vec{s})$, must have the same value at ϕ and $\phi + 2\pi$, we find the eigenfunctions $\Psi_m(r,\phi) = F(r)e^{im\phi}$, where $F(r)$ is an arbitrary function of r, and the eigenvalues are $m = 0, \pm1, \pm2$, etc. If we restrict the function's argument to the unit circle, $r = 1$, we get

$$\Psi_m(\phi) = e^{im\phi} = \cos m\phi + i\sin m\phi.$$

These functions form an orthogonal basis and are known as *circular harmonics*.

All of this is quite familiar to us from our discussion of SU(2) representations on 3D functions. The main difference is that we are now restricted to 2D. The spherical harmonics that we had in 3D become circular harmonics in 2D. Thanks to the lower dimensionality, we can understand these basis functions in an intuitive way. The U(1) group elements represent rotations of a function on a circle. A small element of the u(1) algebra represents the change in this function due to a small rotation (= partial derivative in the angular direction times a small angle). The eigenfunctions of such an algebra element are those functions for which the small change is proportional to the function value itself. If the functions were on a line rather than on a circle, then these eigenfunctions would be exponential functions. Unfortunately, on a circle we cannot find real functions that have the desired "eigenproperty". But we can find *pairs* of functions, such that a small change in one function is proportional to the value of the other function and vice versa. In other words, we can find *complex* eigenfunctions (= pairs of real functions) with imaginary eigenvalues (changes in one function are proportional to the *other* function). These functions are the circular harmonics: $\cos m\phi + i\sin m\phi$. Finally, if we multiply the imaginary eigenvalues with i, we get real numbers that can represent measurement outcomes in quantum mechanics.