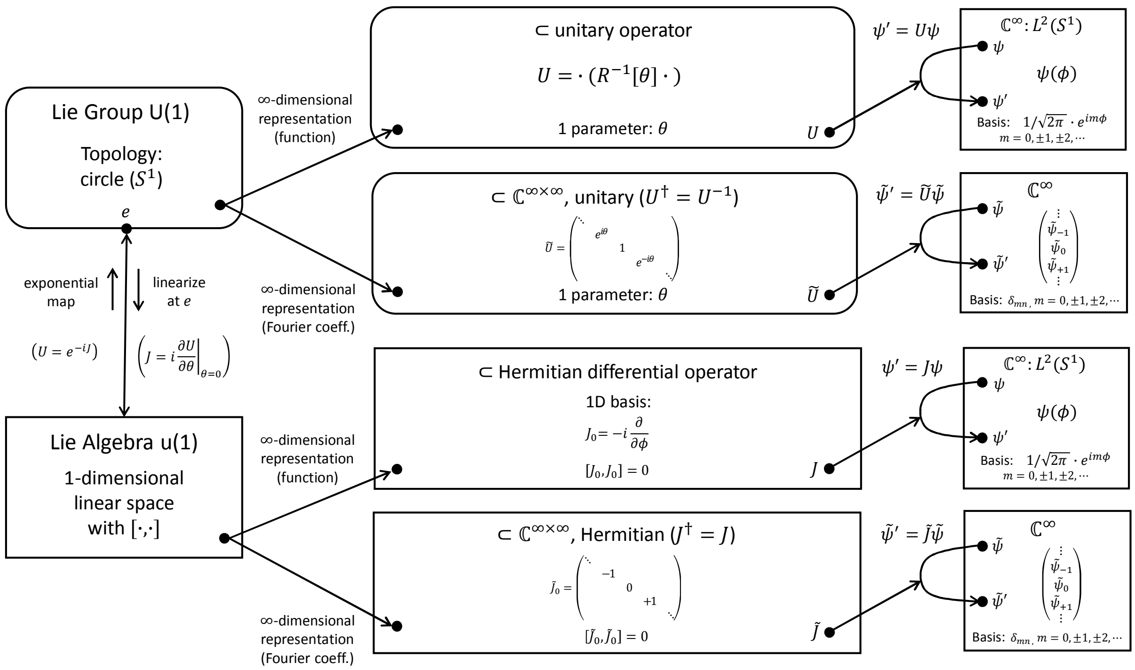
6.6 U(1): Application to Fourier Series and Fourier Coefficients



The circular harmonics that we found in the previous example provide a basis into which we can expand complex functions on the unit circle, $\psi(\phi)$, where ϕ is the angle determining the point on the circle and $\psi(\phi) = \psi(\phi + 2\pi)$. The resulting complex coefficients, $\tilde{\psi}_m$, form an infinite vector, which, like the original periodic function, furnishes a representation of U(1).

Before we start, let's check if the basis functions from the previous example, $\Psi_m(\phi) = e^{im\phi}$, where $m = 0, \pm1, \pm2, \ldots$, are orthogonal and *normalized*. The Hermitian inner product of two such basis functions is $\int_0^{2\pi} \Psi_n^*(\phi)\Psi_m(\phi)\, d\phi = \int_0^{2\pi} e^{-in\phi}e^{im\phi}\, d\phi = \int_0^{2\pi} e^{i(m-n)\phi}\, d\phi$. For $m \neq n$, this integral evaluates to zero, confirming that distinct basis functions are orthogonal. For $m = n$, the integral evaluates to 2π, revealing that the basis functions in their current form are *not* normalized. To make the basis orthonormal, which will become important momentarily, we *redefine* $\Psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$. Now, we have $\int_0^{2\pi} \Psi_n^*(\phi)\Psi_m(\phi)\, d\phi = \delta_{nm}$, as desired. See the upper branch of the diagram.

Our goal is to express the periodic (square-integrable) function $\psi(\phi)$ in terms of these basis functions: $\psi(\phi) = \sum_m \tilde{\psi}_m \Psi_m(\phi) = \cdots + \tilde{\psi}_{-1}\Psi_{-1}(\phi) + \tilde{\psi}_0\Psi_0(\phi) + \tilde{\psi}_{+1}\Psi_{+1}(\phi) + \cdots$. This expansion is known as the *Fourier series* of $\psi(\phi)$ and the $\tilde{\psi}_m$ are the *Fourier coefficients*. The periodic function of a continuous variable, $\psi(\phi)$, is now represented by an infinite-dimensional vector with a discrete index, $\tilde{\psi}_m$. How can we find the coefficients $\tilde{\psi}_m$ for a given function $\psi(\phi)$? Simply projecting the function on the orthonormal basis does the trick! Specifically, taking the Hermitian inner product of $\psi(\phi)$ and the n-th basis function $\Psi_n(\phi)$ gives us the n-th Fourier coefficient: $\int_0^{2\pi} \Psi_n^*(\phi)\psi(\phi)\, d\phi = \int_0^{2\pi} \Psi_n^*(\phi) \sum_m \tilde{\psi}_m \Psi_m(\phi)\, d\phi = \sum_m \tilde{\psi}_m \int_0^{2\pi} \Psi_n^*(\phi)\Psi_m(\phi)\, d\phi = \sum_m \tilde{\psi}_m \delta_{nm} = \tilde{\psi}_n$. In the first step, we expanded the function $\psi(\phi)$ into its Fourier series; in the second step, we pulled the (constant) Fourier coefficients $\tilde{\psi}_m$ out of the integral; in the third step, we made use of the fact that the basis is orthonormal; and in the last step we multiplied out δ_{nm} (= identity matrix).

To summarize, the Fourier series of a periodic function is

$$\psi(\phi) = \sum_{m=0,\pm1,\cdots} \tilde{\psi}_m \Psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{m=0,\pm1,\cdots} \tilde{\psi}_m e^{im\phi}$$

and the Fourier coefficients of that series are

$$\tilde{\psi}_m = \int_0^{2\pi} \psi(\phi)\Psi_m^*(\phi)\, d\phi = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(\phi) e^{-im\phi}\, d\phi.$$

Now, let's investigate the representation that acts on the Fourier coefficients, which are collected in an infinite-dimensional vector. We know that the original function transforms like $\psi'(\phi) = \psi(\phi - \theta)$ when rotated by θ. Calculating the Fourier coefficients of the transformed function tells us how the coefficients transform: $\tilde{\psi}_m' = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(\phi - \theta) e^{-im\phi}\, d\phi = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(\varphi) e^{-im(\varphi+\theta)}\, d\varphi = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(\varphi) e^{-im\varphi} e^{-im\theta}\, d\varphi = \tilde{\psi}_m e^{-im\theta}$. In the first step, we introduced a new variable $\varphi = \phi - \theta$; in the second step, we factored the exponential term such that in the third step we can isolate the original Fourier coefficients $\tilde{\psi}_m$. We conclude that the m-th Fourier coefficient gets multiplied by the phase factor $e^{-im\theta}$ when rotated by θ. We can express this operation as a diagonal ($\infty \times \infty$) matrix acting on the Fourier-coefficient vector: $\tilde{\psi}' = \tilde{U}\tilde{\psi}$. See the lower branch of the diagram.

What about the corresponding Lie algebra? Differentiating $\tilde{\psi}_m e^{-im\theta}$ with respect to θ, setting $\theta = 0$, and multiplying by i reveals $\tilde{\psi}_m' = m\tilde{\psi}_m$. This operation can also be expressed as a diagonal ($\infty \times \infty$) matrix acting on the Fourier-coefficient vector: $\tilde{\psi}' = \tilde{J}\tilde{\psi}$. See the lower branch of the diagram.

Examining the $\tilde{U}$ and $\tilde{J}$ matrices, we see that this representation is the direct sum of *all* irreducible representations of U(1)! We started out with two irreducible representations ($m = \pm 1$) and constructed from them an infinite-dimensional representation on functions. We ended up finding all the irreducible representations of U(1) hiding inside this infinite-dimensional representation.

The periodic function $\psi(\phi)$ can be visualized as a closed curve in the complex plane, ψ, parametrized by the angle $\phi = 0..2\pi$. For fun $\psi(\phi)$ could be chosen to outline a portrait of Joseph Fourier! Such a curve can also be expressed as the Fourier series $\psi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_m \tilde{\psi}_m\, e^{im\phi} = \frac{1}{\sqrt{2\pi}} (\tilde{\psi}_{-5} e^{-i5\phi} + \cdots + \tilde{\psi}_0 + \cdots + \tilde{\psi}_{+5} e^{+i5\phi})$, where we assumed for the purpose of this example that eleven Fourier coefficients are sufficient. For $\phi = 0$ we have $\psi(0) = \frac{1}{\sqrt{2\pi}} \sum_m \tilde{\psi}_m = \frac{1}{\sqrt{2\pi}} \left(\tilde{\psi}_{-5} + \cdots + \tilde{\psi}_0 + \cdots + \tilde{\psi}_{+5}\right)$, which can be visualized as the end point of eleven concatenated pointers (= vectors), each one representing a complex Fourier coefficient. As we sweep ϕ from 0 to 2π, five of the pointers start to rotate clockwise and five of them counterclockwise, each one with its particular integer-valued angular frequency ($m = -5, \ldots, 0, \ldots, +5$). The length of each pointer stays fixed. The end point of this rotating arrangement of pointers magically traces out the closed curve corresponding to $\psi(\phi)$! Watch the beautiful animations at https://www.3blue1brown.com exploiting this process to draw portraits of Joseph Fourier.