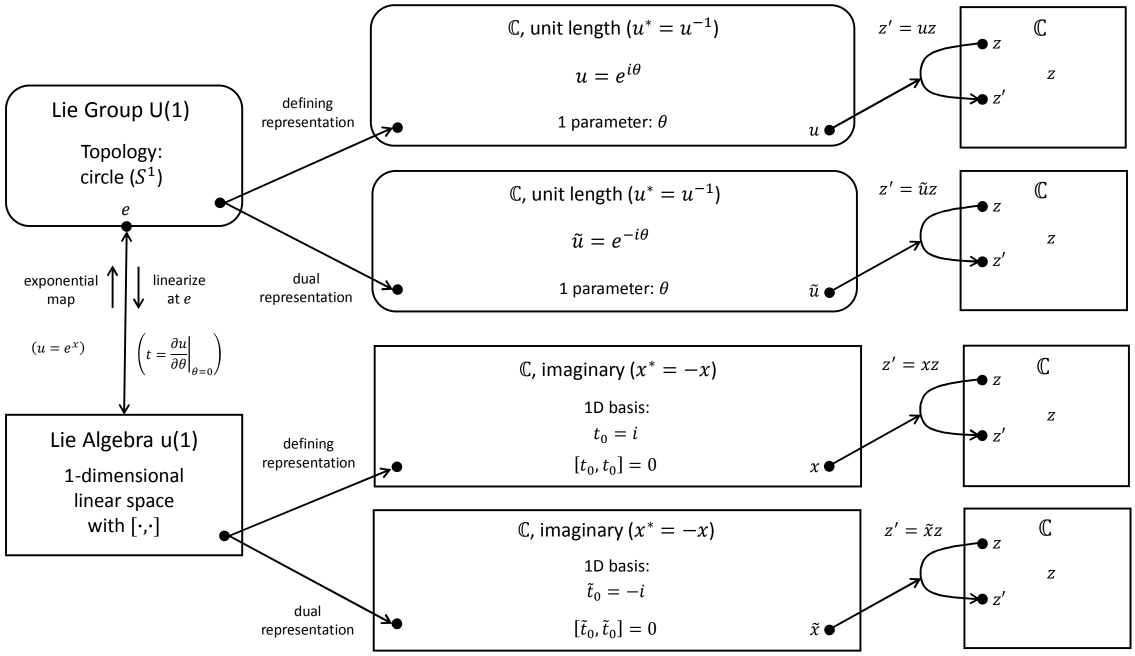
6.2 U(1): Dual, Complex-Conjugate, Adjoint, and Product Representations



When studying SU(2), we discussed dual representations (which act on dual vector spaces), complex-conjugate representations (which act on complex-conjugated vector spaces), the adjoint representation (which acts on the Lie algebra of the defining representation), and tensor-product representations (which act on tensor-product spaces of two representations). What are the corresponding representations for U(1)?

Let's start with the *dual representation*. We know from SU(2) that given a transformation matrix, U, the matrix of the dual representation is given by the transpose of its inverse, U^{-1T}. Applying this operation to the defining representation $e^{i\theta}$, which is shown again in the upper branch of the diagram, yields $\tilde{u} = u^{-1} = e^{-i\theta}$, which is shown in the lower branch. Note that here U is a 1×1 matrix, that is, a scalar, and thus the transpose operation has no effect. In conclusion, the dual of the defining representation is the $k = -1$ representation. More generally, the dual of the k representation is the $-k$ representation.

Unitary transformations satisfy $U^\dagger = U^{-1}$, which can also be written as $U^* = U^{-1T}$. Therefore, the dual of a unitary representation is also the *complex-conjugate representation*. Applied to the defining representation, the complex-conjugate representation is $\tilde{u} = u^* = e^{-i\theta}$.

But there is an important difference between U(1) and SU(2): The defining representation of SU(2) and its complex-conjugate representation were equivalent, that is, there was a similarity transformation (= change of basis of the representation space) that maps one to the other. In contrast, the defining representation of U(1) and its complex-conjugate representation are *inequivalent*, that is, there is *no* similarity transformation relating the two! The same is true for all irreducible representations of U(1). Why? The similarity transformation, $\tilde{u} = sus^{-1}$, in which u and s are both scalars, doesn't do anything: $\tilde{u} = u$! We need at least a 2-dimensional representation for a similarity transformation to do something interesting. In some sense, the lower dimensionality of U(1) makes it more "rigid" than SU(2) preventing a relationship between the k and $-k$ representations.

If a representation is inequivalent to its complex conjugate, like the irreducible representations of U(1), we call it *complex*; if a representation looks complex, but is equivalent to its complex conjugate, like the defining representation of SU(2), we call it *pseudoreal*; finally, if a representation is equivalent to a manifestly real representation, like the 3-dimensional representation of SU(2), we call it *real* [GTNut, II.4].

What is the *adjoint representation* of U(1)? We know from SU(2) that the adjoint representation acts on the Lie-algebra space by conjugation. But because u is a scalar, conjugation is the identity operation: $z' = uzu^{-1} = z$, for any $u = e^{i\theta}$. In other words, the adjoint representation is the *trivial representation* labeled by $k = 0$! Again, the low dimensionality of U(1) results in a "rigidity", which, in this case, keeps the entire tangent space (= Lie algebra) in place. Later, when we discuss gauge theory, we will see that a consequence of the adjoint representation of U(1) being trivial is that the electromagnetic field (or photon) does not carry an electric charge!

Finally, let's look at the *tensor-product representations* of U(1). All irreducible representations of U(1) are one dimensional. Thus, taking the tensor product of any two of them yields again a 1-dimensional representation. How does the product representation act on its representation space? If a first representation acts like $v' = ve^{im\theta}$ and a second representation acts like $w' = we^{in\theta}$, then the product $z = vw$ transforms like $z' = v'w' = ve^{im\theta}\, we^{in\theta} = ze^{i(m+n)\theta}$. Thus, the product representation is the representation labeled by $k = m + n$.