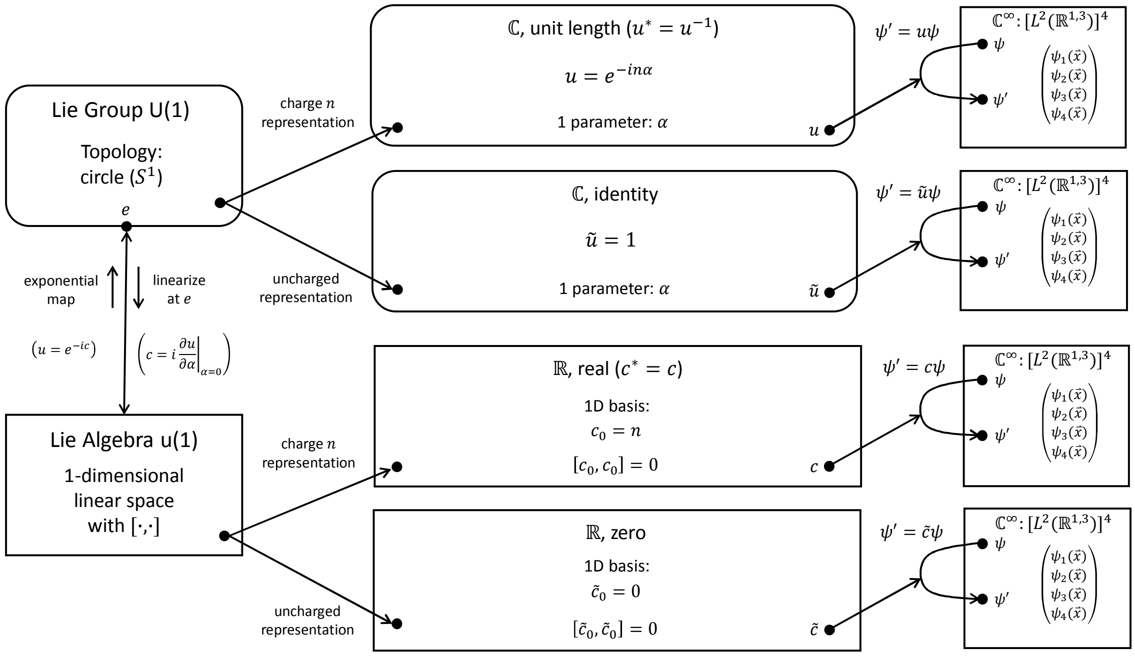
6.7 U(1): Application to the Electron Field; Electric Charge



We now turn to a more abstract but very important application of U(1): phase transformation of an *electron field* and the associated observable, the *electric charge*. But first, what is an electron field?

To describe electrons quantum mechanically, we usually start with a classical point-particle model and then introduce a wave function of the particle configurations to account for the wave properties. Interestingly, there is a more general and more powerful way to do this! We start by regarding electrons as a classical *field* and then introduce a wave function of the field configurations. This may not sound very promising: we are attempting to describe particles with a wave function of a wavy field! But it turns out that we can decompose the field into a superposition of plane waves and then treat each plane wave as a quantum harmonic oscillator. This allows us to interpret the wave function of the field configurations as a collection of particles with their energies and momenta given by the frequencies and wave vectors of the plane waves and the number of particles in each plane wave given by the level of excitation of the quantum harmonic oscillator! This is the idea of quantum field theory in a nutshell [BIU, Idea 9: https://www.youtube.com/watch?v=Dy1LNk_B6IE].

In other words, to arrive at quantum mechanics (QM), we start with a classical particle picture and then add the wave aspect to the particle picture. To arrive at quantum field theory (QFT), we start with a classical field picture and then add the particle aspect to the wave picture. The big advantage of the latter approach is that it can deal with a variable number of particles, as required by the special theory of relativity (particles can convert to energy and vice versa). In QM, the wave function of an n-particle system is defined over a $3n$-dimensional configuration space and can only ever describe n particles. In QFT, the wave function is defined over an infinite-dimensional configuration space and may describe any number of particles including superpositions of different numbers of particles!

Different classical fields yield different particles when quantized. A complex field yields particles and antiparticles; a real field yields only particles; a spinor field yields spin-½ particles; a vector field yields spin-1 particles, etc. The classical electron field is written as $\psi(\vec{x})$, where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is a *Dirac spinor*, which is defined at every point in space time $\vec{x} = (t, \vec{x})^T = (t, x, y, z)^T$ (the harpoon indicates a 4-vector). When quantized, this field yields electrons and positrons, that is, spin-½ particles and antiparticles [QFTGA, Ch. 38.1]. Note that in this context ψ is a classical field, *not* a wave function; ψ is a function of physical space-time, *not* configuration space.

The U(1) representation shown in the diagram (upper branch) acts on the *phase* of the Dirac spinor field: $\psi'(\vec{x}) = e^{-in\alpha}\psi(\vec{x})$, where α is the transformation parameter and $n \in \mathbb{Z}$ labels the representation. Whereas, the spin representations of SU(2), discussed earlier, acted on the individual components of a two-component spinor, this U(1) representation acts on the *overall* phase of the spinor. The generator of this representation (multiplied times i) is n.

Is $e^{-in\alpha}$ a symmetry transformation of the electron field? To find out, let's check if the Dirac action or its associated Lagrangian density, which governs the dynamics of the electron field, remains invariant. Stripping the Dirac Lagrangian density down to a "toy form", which ignores the spinor nature of the field and all but one space directions, we get $\mathcal{L} = i\psi^* \, \partial\psi/\partial t - i\psi^* \, \partial\psi/\partial x - m\psi^*\psi$, where $\psi(t, x)$ is now just a complex function of time and one space dimension. (For the full Dirac Lagrangian density, see the Appendix "Decoding the Dirac Equation".) Applying our U(1) transformation to ψ, we see that our toy Lagrangian density remains invariant: $\mathcal{L}' = ie^{in\alpha}\psi^* \, \partial(e^{-in\alpha}\psi)/\partial t - ie^{in\alpha}\psi^* \, \partial(e^{-in\alpha}\psi)/\partial x - me^{in\alpha}\psi^* e^{-in\alpha}\psi = \mathcal{L}$. It turns out that the same is true for the full Dirac Lagrangian density [PfS, Ch. 7.1.1]. Not surprisingly, it doesn't matter what phase of the electron field we call zero!

What is the conserved quantity associated with this U(1) symmetry? Noether's theorem for field theories says that given a field $\psi_i(\vec{x})$ and a global linear symmetry transformation $S_{ij}(\alpha)$ acting on it, the quantity $Q = \int \sum_{ij} \pi_i(\vec{x}) G_{ij} \psi_j(\vec{x}) \, d^3x$ is conserved, where $\pi_i(\vec{x}) = \partial\mathcal{L}/\partial(\partial\psi_i/\partial t)$ are the (time-like) canonical momentum densities conjugate to $\psi_i(\vec{x})$ and G_{ij} is the generator matrix of the symmetry transformations $S_{ij}(\alpha)$. For our toy Dirac Lagrangian density, the sum under the integral evaluates to $i\psi^*(t, x)[-in]\psi(t, x) + 0[in]\psi^*(t, x)$ and thus the conserved quantity becomes $Q = n \int \psi^*\psi \, d^3x$. For the full Dirac Lagrangian density, the same procedure yields $Q = n \int \psi^\dagger\psi \, d^3x$ [PfS, Ch. 7.1.6] (see the Appendix "Noether's Theorem for Field Theories; Internal Symmetries"). Thus, the conserved quantity is an integer multiple of the square norm of the electron field! But what does this mean?

The electron field is a spin-½ field and because of the Pauli exclusion principle it never looks like a classical field. Instead, it looks like a collection of particles. It turns out that when replacing the classical field with a quantum field in the above expression, the conserved quantity becomes the *number* of electrons minus the number of positrons: $n_e - n_p$, or more generally, the number of particles minus the number of antiparticles [QFTGA, Ch. 38.1]. (Strictly speaking, we should derive the conserved quantity for a quantum field from scratch because we cannot rely on Noether's theorem from classical physics [Mindscape Q&A, 11/23].) Since electrons carry the charge $-e$ and positrons the charge $+e$, the conserved quantity can also be expressed as the *electric charge* $q = ne$, where $n = n_p - n_e$ and e is the elementary charge. Note that the charge observable $n = q/e$ is the generator of this U(1) symmetry.

The electric charge couples the particles to the electromagnetic field. Next, we need to bring the electromagnetic field into the picture!