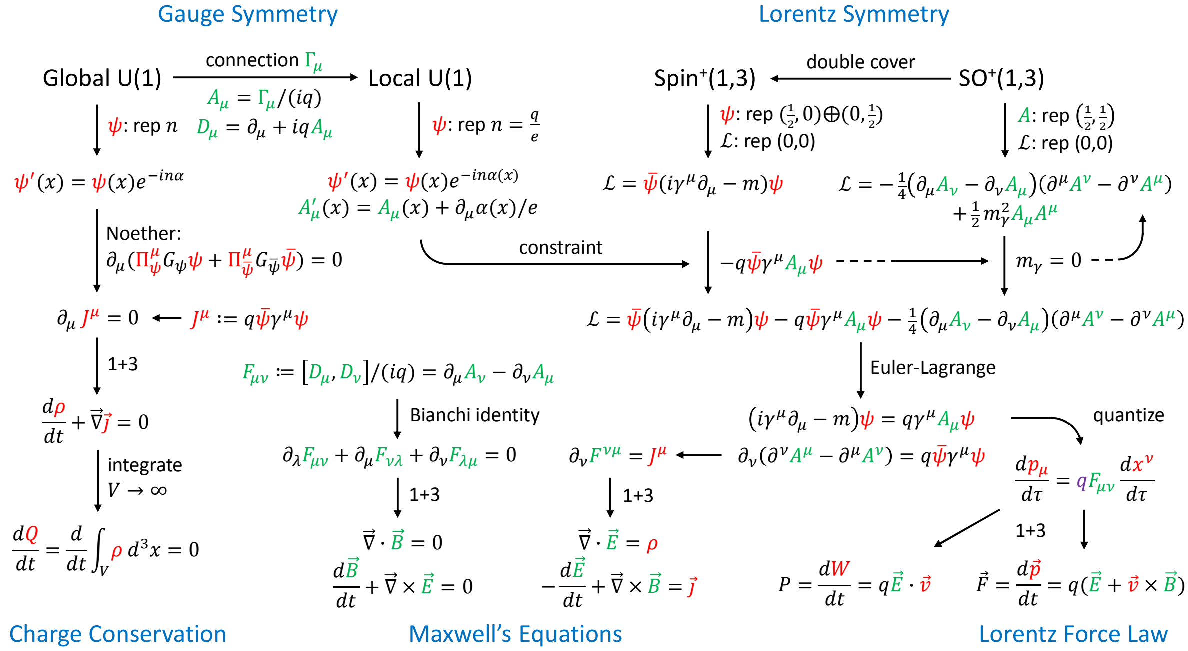
9.21 From Lorentz and Gauge Symmetry to Maxwell's Equations



In the following, we sketch the logical steps that lead from Lorentz symmetry and U(1) gauge symmetry to Maxwell's equations of electrodynamics.

Lorentz symmetry (= space-time symmetry according to the special theory of relativity), which is described by the group SO⁺(1,3), permits a 4-vector field, $A_\mu(x)$, that evolves in time according to the Proca Lagrangian $\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}m_\gamma^2 A_\mu A^\mu$. Furthermore, the double cover of SO⁺(1,3), namely Spin⁺(1,3), permits a quantum-mechanical spinor field, $\psi(x)$, that evolves in time according to the Dirac Lagrangian $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$. When quantized, the latter field yields matter particles, which may be electrically charged, such as electrons and positrons. The $A_\mu(x)$ and $\psi(x)$ fields furnish representations of the respective symmetry groups and the two Lagrangians are invariant under the same groups (= furnish the trivial representation). See the top-right section of the diagram.

The complex spinor field $\psi(x)$ respects the global U(1) symmetry $\psi'(x) = \psi(x)e^{-in\alpha}$, that is, it doesn't matter what phase we call zero. According to Noether's theorem, this symmetry implies the divergence-free current density $\partial_\mu(n\bar\psi\gamma^\mu\psi\,) = 0$ (generators: $G_\psi = -in$, $G_{\bar\psi} = in$; $\Pi^\mu_\psi = i\bar\psi\gamma^\mu$, $\Pi^\mu_{\bar\psi} = 0$). When multiplied by the elementary charge e, this current can be identified with the electrical current density $J^\mu = q\bar\psi\gamma^\mu\psi$, where $q = ne$. Splitting the 4-vector into a time and space part, $J^\mu = (\rho, \vec j)^T$, the condition $\partial_\mu J^\mu = 0$ can be rewritten as a continuity equation: $\partial_t\rho + \vec\nabla\vec j = 0$. Integrating the time-like component ρ over all of space, applying Gauss' theorem, and assuming that $\vec j(x)$ goes to zero at infinity, we find the conserved scalar quantity $Q = q\int\psi^\dagger\,\psi d^3x$. After quantization, this represents the total electric charge of the particles associated with the field. See the left side of the diagram.

Furthermore, the complex spinor field exhibits the *local* U(1) symmetry $\psi'(x) = \psi(x)e^{-in\alpha(x)}$, if we introduce the appropriate connection field $\Gamma_\mu(x)$ that defines "parallel transport" for the spinor field's phase values. This connection field can be identified with the electromagnetic potential field: $A_\mu(x) =$

$\Gamma_\mu(x)/(iq)$. The introduction of a connection requires us to upgrade the ordinary derivative ∂_μ to the covariant derivative $D_\mu = \partial_\mu + iqA_\mu(x)$. Crucially, *local* U(1) transformations must act simultaneously on the spinor field *and* the connection field: $\psi'(x) = \psi(x)e^{-in\alpha(x)}$ and $A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x)/e$. Like the spinor field, the newly introduced 4-vector field $A_\mu(x)$ lives in space-time and thus must furnish a representation of SO+(1,3) and evolve according to a Lagrangian that is invariant under the same group. See the top-center section of the diagram.

The introduction of a (nontrivial) connection also requires us to upgrade the spinor field $\psi(x)$ from a function of space-time to a section of a fiber bundle of space-time. We can think of the spinor field $\psi(x)$ as living in the fiber-bundle space "glued together" by the electromagnetic potential field $A_\mu(x)$. The curvature of the bundle, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, can be identified with the electromagnetic field-strength tensor, which is invariant under local U(1) transformations. Splitting this 4-tensor into time and space parts yields the electric field $\vec{E}$ and the magnetic field $\vec{B}$:

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad \vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \qquad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}.$$

The local U(1) symmetry, which acts simultaneously on $\psi(x)$ and $A_\mu(x)$, constrains the Lagrangian that governs the two fields. It turns out that an interaction term needs to be added to the free terms, the simplest version of which is $-q\bar{\psi}\gamma^\mu A_\mu$, and that the A_μ field needs to be dispersion free, $m_\gamma = 0$ (the kinetic part of the Proca Lagrangian already satisfies the local U(1) symmetry [PfS, Ch. 6.4, p. 129]). The resulting Lagrangian is $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$ [QFTGA, Ch. 38.4; PfS, Ch. 7.1.3]. It defines the dynamics of quantum electrodynamics (QED).

To obtain the equations of motion from the above Lagrangian, we vary its degrees of freedom (= fields) and determine where the action is stationary: $\delta\int\mathcal{L}\,d^4x = 0$. This procedure results in the Euler-Lagrange equations. Varying A_μ yields the equation of motion $\partial_\nu F^{\nu\mu} = J^\mu$. Splitting this equation into a time and space part yields the inhomogeneous Maxwell equations. The homogeneous Maxwell equations follow from the facts that $F_{\mu\nu}$ is the exterior derivative of A_μ and that taking the exterior derivative a second time yields zero (= Bianchi identities). See the bottom-center section of the diagram.

Varying $\bar{\psi}$ yields the equation of motion $(i\gamma^\mu\partial_\mu - m)\psi = q\gamma^\mu A_\mu\psi$. This equation can be rewritten as $(i\gamma^\mu D_\mu - m)\psi = 0$, which is just the Dirac equation with a covariant derivative. (Varying ψ does not yield an independent equation of motion.) To derive the Lorentz force law, which refers to charged *particles*, we need to quantize the spinor field and express it in terms of particle positions and momenta, $\vec{x}(t)$ and $\vec{p}(t)$. This is not an easy task! One approach is to take the free Schroedinger equation, $i\partial_t\psi = -1/(2m)\,\vec{\nabla}^2\psi$, as the nonrelativistic limit of the Dirac equation, where $\psi(x)$ now represents a single-particle wave function, then upgrade it with a covariant derivative, $i(\partial_t + iqV)\psi = -1/(2m)(\vec{\nabla} - iq\vec{A})^2\psi$, and calculate the classical expectation values [PfS, Ch. 11.2]. The relativistic equation of motion for a particle in an electromagnetic field, $dp_\mu/d\tau = qF_{\mu\nu}(dx^\nu/d\tau)$, splits into a time and space part, where the latter simplifies to the Lorentz force law and the former to an unnamed law for the particle's power [TM, Vol. 3, Ch. 6.3]. See the bottom-right section of the diagram.