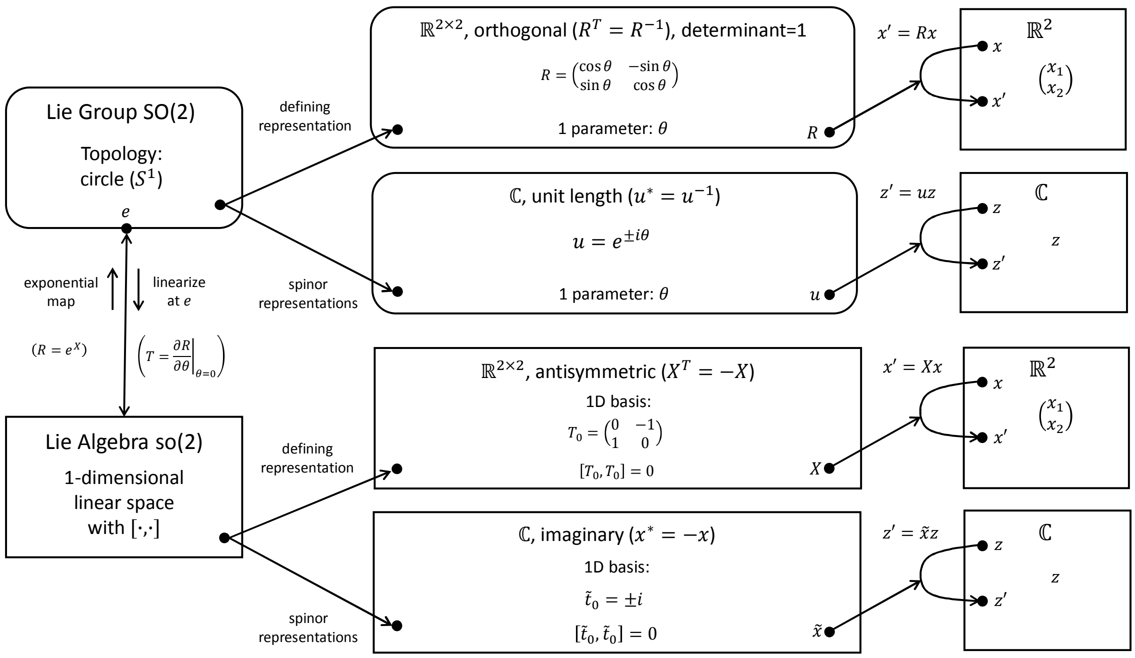
6.13 SO(2): Spinor Representations; Spin(2) = SO(2)



In the previous example, we showed that SO(2) and U(1) are isomorphic. Hence, all representations of U(1) are also representations of SO(2). This includes the 1-dimensional representations of the form $u = e^{ik\theta}$, where $k = 0, \pm1, \pm2, \ldots$ But these representations act on a *complex* variable: they are *spinor representations* of SO(2)! The lowest two spinor representations (with $k = \pm1$), usually designated S^+ and S^-, are shown in the lower branch of the diagram [GTNut, Ch. VII.1, p. 410]. For comparison, the defining (2-dimensional) representation of SO(2) is shown again in the upper branch.

These 1-dimensional spinor representations are the *irreducible representations* of SO(2). Remember that the irreducible representations of a commuting (= Abelian) group, such as SO(2), must be one dimensional [QTGR, Ch. 2.1].

We know from our discussion of U(1) that the S^+ and S^- representations are inequivalent, that is, there is no similarity transformation taking one to the other. Moreover, they have opposite handedness or *chirality* because they are related by the substitution $\theta \to -\theta$.

We first encountered spinor representations when discussing SO(3). There, the so(3) algebra had even-dimensional spinor representations, with the lowest one being two dimensional. Furthermore, when discussing SO(4) and later SO⁺(1,3), we found that the algebra had spinor representations, with the lowest ones being the 2-dimensional representations (½, 0) and (0, ½). These representations were irreducible, inequivalent, and had opposite chirality, just like the S^+ and S^- representations of SO(2).

But there is something funny going on: Exponentiating the spinor representations of so(3) led us to representations of Spin(3), the double cover of SO(3). Similarly, exponentiating the spinor representations of so(4) led us to representations of Spin(4), the double cover of SO(4). So, we would expect that exponentiating the spinor representations of so(2) takes us to representations of Spin(2), the double cover of SO(2). But the $e^{\pm i\theta}$ transformations, shown in the diagram, do *not* double cover SO(2): a 360° rotation doesn't take z to $-z$.

It turns out that Spin(2) and SO(2) are isomorphic [nLab, ncatlab.org]! Topologically, both of them are circles. So, taking the exponential map of so(2) does take us to Spin(2), but it's *not* a new group, it's just SO(2) again. (In contrast, the *universal* cover of SO(2) is a new group, namely $\mathbb{R}$.)

For SO(4) and SO⁺(1,3) we were able to construct a nonchiral spinor representation by taking the direct sum of its left- and right-chiral (half) spinor representations: (½, 0) ⊕ (0, ½). Analogously, we can construct a nonchiral spinor representation of SO(2) by taking the direct sum of its two chiral (half) spinor representations, $S^+ \oplus S^-$:

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Flipping the chirality of this representation by substituting $\theta \to -\theta$ swaps the matrix components on the diagonal. Now, this is a similarity transformation, $U \to SUS^{-1}$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

and therefore this 2-dimensional spinor representation is nonchiral. Moreover, it is equivalent to the defining representation of SO(2):

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Note how different this is from spinors in four dimensions: the 4-dimensional Dirac spinor representation and the defining 4-vector representation were very different animals!

For another perspective on the spinor representation of SO(2), let's construct it by starting from the Clifford algebra Cliff(2, 0) [GTNut, Ch. VII.1, p. 410]. We need two gamma matrices that anticommute and square to one: $\gamma_1\gamma_2 = -\gamma_2\gamma_1$, $\gamma_1^2 = I$, and $\gamma_2^2 = I$, where I is the identity matrix. The first two Pauli matrices fit the bill:

$$\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Then, we construct from those the generator of the spinor representation as $\frac{1}{2}\gamma_1\gamma_2$. The corresponding transformation is $U = e^{\gamma_1\gamma_2\theta/2}$:

$$\tfrac{1}{2}\gamma_1\gamma_2 = \tfrac{1}{2}i\sigma_3 = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad U = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

Now this looks like an honest spinor representation! But because Spin(2) = SO(2), it is equivalent to the $S^+ \oplus S^-$ representation that we constructed earlier.