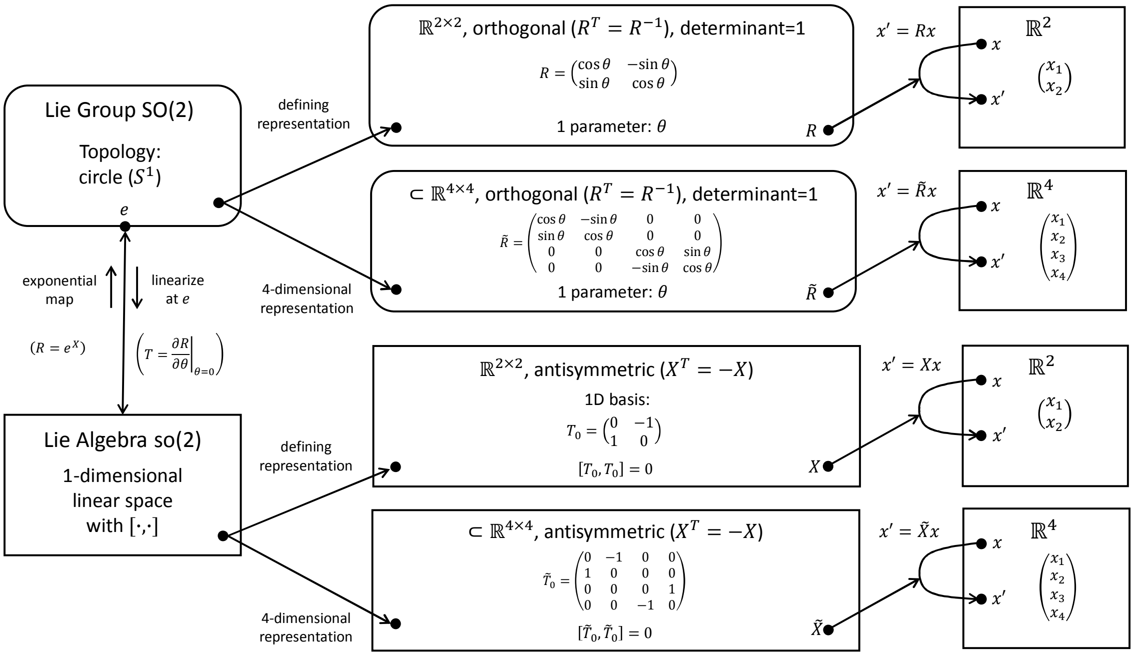
6.12 SO(2): The Group of Rotations in 2-Dimensional Euclidean Space; SO(2) = U(1)



Let's move on to the group SO(2), or rather its defining (2-dimensional) representation, which consists of all 2×2 orthogonal matrices R, with determinant one. As we know from our discussion of SO(3) and SO(4), orthogonal means that R is real and satisfies $R^T R = I$. Such transformations preserve the Euclidean inner product of two vectors. As a consequence, lengths of vectors and angles between vectors remain invariant. Therefore, the orthogonal transformations are simply rotations and/or reflections about the origin. The additional constraint "determinant one", that is, $\det(R) = R_{11}R_{22} - R_{12}R_{21} = 1$, eliminates the reflections, leaving us with only the *proper* 2D rotations. SO(2) is a one-dimensional group, commonly parametrized by the rotation angle.

Like for SO(3) and SO(4), the S in SO(2) stands for *special* and refers to the determinant-one constraint. The larger group O(2) has the topology of two unconnected circles, whereas SO(2) is just a single circle, S^1. However, unlike for SO(3), we cannot write O(2) = SO(2)×$\mathbb{Z}_2$. It turns out that for even-dimensional orthogonal groups, we need to replace the direct product by the semidirect product: O(2) = SO(2)⋊ $\mathbb{Z}_2$! (See https://math.stackexchange.com/questions/1055363/is-o2-really-not-isomorphic-to-so2-times-1-1?rq=1 as well as our example for the dihedral group of order three, D_3.)

The upper branch of the diagram shows the transformation matrix $R(\theta)$ of the defining representation of SO(2), where θ is the angle of rotation. We recognize this matrix as the 2-dimensional real representations of U(1). SO(2), just like U(1), has only one basis generator, T_0, which we find by taking the derivative of $R(\theta)$ and evaluating the result at $\theta = 0$.

In fact, U(1) and SO(2) are isomorphic! To show this isomorphism we need to establish a dictionary translating between the two groups, just like we did for Sp(1) and SU(2). If we identify the complex number $z = a + ib$ with the real 2×2 matrix

$$M = \begin{pmatrix}a & -b \\ b & a\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} a + \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix} b,$$

complex multiplication and matrix multiplication do the same thing! In other words, the matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ behaves like 1 and the matrix $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ behaves like i. For example, $1^2 = 1$ translates to $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, similarly, $i^2 = -1$ translates to $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = -\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Both translations are true statements. Next, we identify the unit-length complex number $u = e^{i\theta} = \cos\theta + i\sin\theta$, which is an element of U(1), with the matrix

$$R = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\cos\theta + \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\sin\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}.$$

This is exactly the SO(2) matrix shown in the upper branch of the diagram!

With this correspondence established, we can translate a U(1) complex number acting on an arbitrary complex number, $z' = uz$, to an SO(2) matrix R acting on a matrix M of the form defined above, $M' = RM$. There is just one small issue: we would like the SO(2) matrix to act on a column *vector*, $x' = Rx$, not on a matrix, $M' = RM$. To resolve this discrepancy, we keep only the first column of M, that is, we translate z to the vector $x = \begin{pmatrix}a\\b\end{pmatrix}$. The second column of M is redundant.

The same dictionary also works for translating between the two Lie algebras. For example, the u(1) generator $t_0 = i$ translates to the so(2) generator $T_0 = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$.

Interestingly, the relationship between SO(2) and U(1) is somewhat different from that between SO(3) and SU(2). Whereas SO(2) and U(1) are fully isomorphic as demonstrated above, for SO(3) and SU(2) only their *algebras* were isomorphic. The groups SO(3) and SU(2) themselves were *not* isomorphic: in fact, the latter double covers the former.

SO(2) has infinitely many representations. For starters, we can use our dictionary to translate all the irreducible representations of U(1): $u = e^{ik\theta}$, where $k = 0, \pm 1, \pm 2, \ldots$, to representations of SO(2): $R = \begin{pmatrix}\cos k\theta & -\sin k\theta\\\sin k\theta & \cos k\theta\end{pmatrix}$. Moreover, we can combine those into larger representations using the direct sum. An example in which we combined two 2-dimensional representations (one with $k = +1$ and one with $k = -1$) into a 4-dimensional one is shown in the lower branch of the diagram.

The defining representation of U(1) was irreducible; is the defining representation of SO(2) also irreducible? If we insist on using only real numbers, we can't break it up into smaller representations, but if we allow complex numbers, we can. We'll see in the next example that SO(2) has 1-dimensional *spinor representations* and these are the irreducible ones!