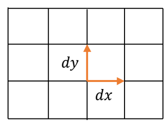
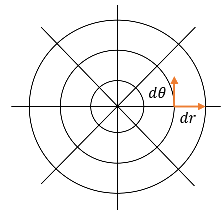
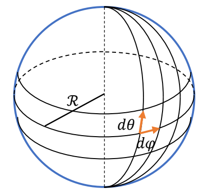
9.19 Metric, Connection, and Curvature in 2D Riemannian Geometry

Space:	2D Euclidean (flat)	2D Euclidean (flat)	2D Spherical
Coordinates:	Cartesian (x and y)	Polar (radius r and angle θ)	Colatitude θ and longitude φ
			
Metric:	$ds^2 = dx^2 + dy^2$	$ds^2 = dr^2 + r^2 d\theta^2$	$ds^2 = \mathcal{R}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$	$\mathcal{R}^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$
Levi-Civita Connection:			
$(\Gamma^i{}_{jm}) = \begin{pmatrix} \begin{pmatrix} \Gamma^1{}_{11} & \Gamma^1{}_{21} \\ \Gamma^2{}_{11} & \Gamma^2{}_{21} \end{pmatrix} \\ \begin{pmatrix} \Gamma^1{}_{12} & \Gamma^1{}_{22} \\ \Gamma^2{}_{12} & \Gamma^2{}_{22} \end{pmatrix} \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix} \\ \begin{pmatrix} 0 & -r \\ 1/r & 0 \end{pmatrix} \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \cot\theta \end{pmatrix} \\ \begin{pmatrix} 0 & -\sin\theta\cos\theta \\ \cot\theta & 0 \end{pmatrix} \end{pmatrix}$
Riemann Curvature:			
$(R^i{}_{jmn}) = \begin{pmatrix} \begin{pmatrix} R^1{}_{111} & R^1{}_{211} \\ R^2{}_{111} & R^2{}_{211} \end{pmatrix} & \begin{pmatrix} R^1{}_{112} & R^1{}_{212} \\ R^2{}_{112} & R^2{}_{212} \end{pmatrix} \\ \begin{pmatrix} R^1{}_{121} & R^1{}_{221} \\ R^2{}_{121} & R^2{}_{221} \end{pmatrix} & \begin{pmatrix} R^1{}_{122} & R^1{}_{222} \\ R^2{}_{122} & R^2{}_{222} \end{pmatrix} \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$	$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \sin^2\theta \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\sin^2\theta \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$

To build up our intuition about connection and curvature in electromagnetic gauge theory, we study their counterparts in 2D Riemannian geometry. We compare a Euclidean space with Cartesian coordinates, a Euclidean space with polar coordinates, and a spherical space with spherical coordinates.

Euclidean Space with Cartesian Coordinates. In this case, the length of a small line element ds is calculated from its coordinate differentials dx and dy as $ds^2 = dx^2 + dy^2$. Comparing this expression with the general form $ds^2 = g_{xx}dx^2 + (g_{xy} + g_{yx})dxdy + g_{yy}dy^2$, we find that the *metric tensor* g_{ij} equals the identity matrix, $g_{ij} = \delta_{ij}$, reflecting the fact that the basis vectors are position independent and orthonormal. Parallel transporting a differential vector $(dx, dy)^T$ in this coordinate system does *not* change its components and hence the *Levi-Civita connection* $\Gamma^i{}_{jm}$ is zero. Furthermore, the *Riemann curvature tensor* $R^i{}_{jmn}$ is also zero, as expected for a flat Euclidean space.

Euclidean Space with Polar Coordinates. Now, the length of a line element is $ds^2 = dr^2 + r^2 d\theta^2$, where r is the radial and θ is the angular coordinate. The corresponding metric tensor is shown in the diagram. While the local basis vectors are still orthogonal, they now rotate from point to point and the length of the angular basis vector is position dependent. Parallel transporting a small differential vector $(dr, d\theta)^T$ in this coordinate system *does* change its components in a surprisingly complicated way. Hence the Levi-Civita connection $\Gamma^i{}_{jm}$ is nonzero. We can find its components, the so-called *Christoffel symbols*, from geometric considerations (see the Appendix "Christoffel Symbols for Polar Coordinates") or we can calculate them from the metric tensor using the formula $\Gamma^k{}_{jm} = \frac{1}{2} g^{ki}(\partial_m g_{ij} + \partial_j g_{im} - \partial_i g_{jm})$ [TM, Vol. 4, Ch. 3, p. 107]:

$$\begin{pmatrix} \begin{pmatrix} \Gamma^r{}_{rr} & \Gamma^r{}_{\theta r} \\ \Gamma^\theta{}_{rr} & \Gamma^\theta{}_{\theta r} \end{pmatrix} \\ \begin{pmatrix} \Gamma^r{}_{r\theta} & \Gamma^r{}_{\theta\theta} \\ \Gamma^\theta{}_{r\theta} & \Gamma^\theta{}_{\theta\theta} \end{pmatrix} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \left[\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2r \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2r & 0 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 2r \\ 0 & 0 \end{pmatrix} \end{pmatrix} \right] = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix} \\ \begin{pmatrix} 0 & -r \\ 1/r & 0 \end{pmatrix} \end{pmatrix}.$$

Note the symmetry ${\Gamma^i}_{jm} = {\Gamma^i}_{mj}$ (blue and purple). The *covariant derivative* is a derivative that takes the twisting and turning of the local basis vectors into account. Given the vector field $v^i(x)$, the covariant derivative in the m-th direction is $D_m v^i = \partial_m v^i + {\Gamma^i}_{jm} v^j$ [TM, Vol. 4, Ch. 4].

The Riemann curvature tensor ${R^i}_{jmn}$ measures the mismatch between taking the covariant derivative first in the n-th and then in the m-th direction versus doing the same in the reverse order: ${R^i}_{jmn} v^j = D_m D_n v^i - D_n D_m v^i$. In other words, it is the commutator of two covariant derivatives: $[D_m, D_n] v^i$. Plugging in the formula for the covariant derivative, we find the Riemann curvature tensor in terms of the connection: ${R^i}_{jmn} = \partial_m {\Gamma^i}_{jn} - \partial_n {\Gamma^i}_{jm} + {\Gamma^i}_{km} {\Gamma^k}_{jn} - {\Gamma^i}_{kn} {\Gamma^k}_{jm}$ [TM, Vol. 4, Ch. 3, p. 116, different index convention]. For the case of polar coordinates, we find

$$\begin{pmatrix} \begin{pmatrix} {R^r}_{rrr} & {R^r}_{\theta rr} \\ {R^\theta}_{rrr} & {R^\theta}_{\theta rr} \end{pmatrix} & \begin{pmatrix} {R^r}_{rr\theta} & {R^r}_{\theta r\theta} \\ {R^\theta}_{rr\theta} & {R^\theta}_{\theta r\theta} \end{pmatrix} \\ \begin{pmatrix} {R^r}_{r\theta r} & {R^r}_{\theta\theta r} \\ {R^\theta}_{r\theta r} & {R^\theta}_{\theta\theta r} \end{pmatrix} & \begin{pmatrix} {R^r}_{r\theta\theta} & {R^r}_{\theta\theta\theta} \\ {R^\theta}_{r\theta\theta} & {R^\theta}_{\theta\theta\theta} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1/r^2 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1/r^2 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1/r^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ -1/r^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/r^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1/r^2 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/r^2 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1/r^2 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix},$$

which evaluates to zero, as expected for a flat Euclidean space.

Spherical Space with Spherical Coordinates. Now, the length of a line element is $ds^2 = \mathcal{R}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$, where θ is the colatitude (= latitude measured from the south pole), φ is the longitude, and $\mathcal{R}$ is the (fixed) radius of the sphere. Parallel transporting a small differential vector $(d\theta, d\varphi)^T$ in this coordinate system changes its components. The relevant connection can be derived from the metric:

$$\begin{pmatrix} \begin{pmatrix} {\Gamma^\theta}_{\theta\theta} & {\Gamma^\theta}_{\varphi\theta} \\ {\Gamma^\varphi}_{\theta\theta} & {\Gamma^\varphi}_{\varphi\theta} \end{pmatrix} \\ \begin{pmatrix} {\Gamma^\theta}_{\theta\varphi} & {\Gamma^\theta}_{\varphi\varphi} \\ {\Gamma^\varphi}_{\theta\varphi} & {\Gamma^\varphi}_{\varphi\varphi} \end{pmatrix} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{pmatrix} \left[\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2\sin\theta\cos\theta \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2\sin\theta\cos\theta & 0 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 2\sin\theta\cos\theta \\ 0 & 0 \end{pmatrix} \end{pmatrix} \right] = \begin{pmatrix} \begin{pmatrix} 0 & {\color{blue}0} \\ 0 & {\color{purple}\cot\theta} \end{pmatrix} \\ \begin{pmatrix} {\color{blue}0} & -2\sin\theta\cos\theta \\ {\color{purple}\cot\theta} & 0 \end{pmatrix} \end{pmatrix}.$$

This is a nontrivial connection that cannot be made zero everywhere by a change of coordinates. The Riemann curvature tensor derived from this connection is shown in the diagram. Its nonzero value implies that the sphere cannot be flattened without stretching or compressing it. Note the antisymmetry ${R^i}_{jmn} = -{R^i}_{jnm}$ (red and green). Incidentally, contracting the first and the third index of the Riemann tensor yields the Ricci tensor $R_{jn} = {R^i}_{jin} = \begin{pmatrix} R_{\theta\theta} & R_{\theta\varphi} \\ R_{\varphi\theta} & R_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$. Raising one index of the Ricci tensor and contracting again yields the Ricci scalar $R = g^{nj} R_{jn} = 2/\mathcal{R}^2$.

Comparison of Riemannian Geometry with Electromagnetic Gauge Theory. Two major differences stand out: (i) In the case of geometry, the parallel-transported object (= small differential vector) lives in the *tangent space* of the manifold (e.g., a sphere) in which it is moved around. In the case of electromagnetism, the parallel-transported object (= complex value of a charged matter field) lives in an *internal space* that is unrelated to the manifold (= Minkowski space) in which it is moved around. Because of this difference, there is no analog of the metric (or geodesics) in electromagnetic gauge theory. But analogs of the connection and the curvature do exist: they are the electromagnetic potential, A_μ, and the electromagnetic field strength, $F_{\mu\nu}$. (ii) In the case of geometry, the parallel-transported object is a vector, whereas in the case of electromagnetism, it is a (complex) scalar. Hence, in electromagnetic gauge theory, the connection is not a vector of matrices but just a vector, ${\Gamma^i}_{jm} \to \Gamma_\mu = iqA_\mu$, and the curvature tensor is not a matrix of matrices but just a matrix, ${R^i}_{jmn} \to R_{\mu\nu} = iqF_{\mu\nu}$.