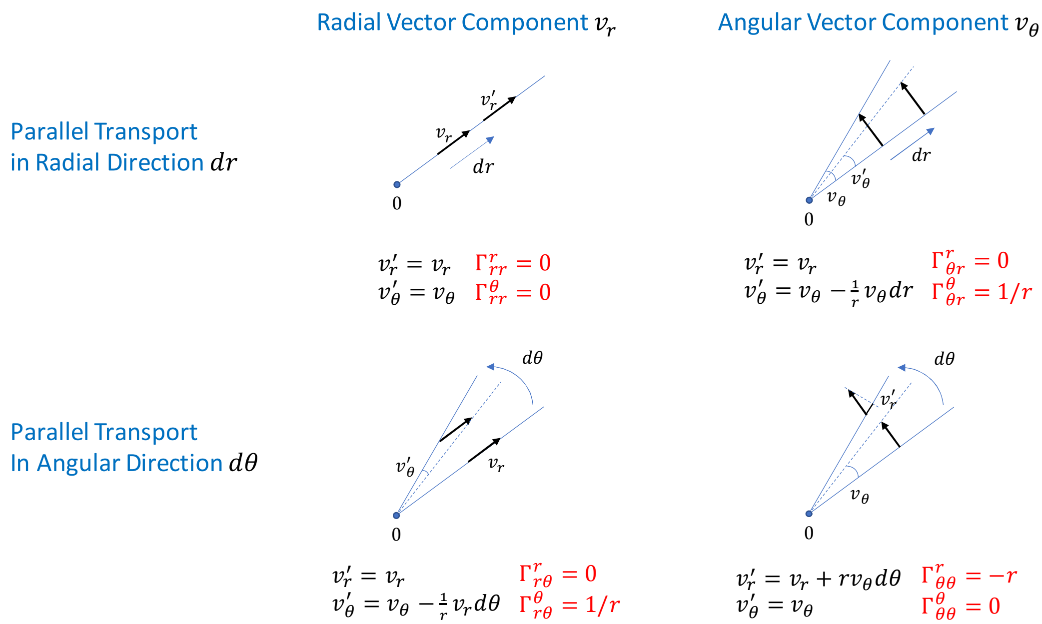
9.20 Christoffel Symbols for Polar Coordinates



We use an intuitive geometric method to determine the Christoffel symbols (= components of the Levi-Civita connection) for a 2D Euclidean space with *polar coordinates*. This exercise clarifies how a connection characterizes the twisting and turning of the coordinates that underly the mathematical description of a geometric vector. Incidentally, for a Euclidean space with *Cartesian coordinates* all Christoffel symbols are zero (because the coordinates are straight) and the entire concept of a connection is not needed. But Cartesian coordinates exist only for flat spaces; if we want to describe curved spaces, curved coordinates together with a non-trivial connection become unavoidable!

The Christoffel symbols Γ^i_{jm} tell us how the components of a small vector change when we parallel transport it from one point to a neighboring point. Let's call this small vector $\vec{v}$ ($d\vec{v}$ would be more appropriate, but to keep the notation simple we drop the d). It has two components: a radial one and an angular one: $\vec{v} = (v_r, v_\theta)^T$. This small vector is located at the point $\vec{x} = (r, \theta)^T$, thus we write $\vec{v}(\vec{x})$. Our objective is to parallel transport this vector to the neighboring point $\vec{x} + d\vec{x}$, where $d\vec{x} = (dr, d\theta)^T$. At its destination, we write the vector as $\vec{v}'(\vec{x} + d\vec{x})$, where $\vec{v}' = (v_r', v_\theta')^T$. With these definitions in place, we can write the parallel-transported vector as a function of the original vector and the transport direction in the following general form:

$$\vec{v}'(\vec{x} + d\vec{x}) = \vec{v}(\vec{x}) - [\Gamma(\vec{x}) \cdot d\vec{x}]\, \vec{v}(\vec{x}).$$

A linear relationship is sufficient because we are transporting to a point that is infinitesimally close. The minus sign is just a convention. Rewriting the above parallel-transport equation in component form reveals that the connection $\Gamma(\vec{x})$ is a column vector with two 2×2 matrices as its components:

$$\begin{pmatrix} v_r' \\ v_\theta' \end{pmatrix} = \begin{pmatrix} v_r \\ v_\theta \end{pmatrix} - \left[\begin{pmatrix} \Gamma^r_{rr} & \Gamma^r_{\theta r} \\ \Gamma^\theta_{rr} & \Gamma^\theta_{\theta r} \end{pmatrix} dr + \begin{pmatrix} \Gamma^r_{r\theta} & \Gamma^r_{\theta\theta} \\ \Gamma^\theta_{r\theta} & \Gamma^\theta_{\theta\theta} \end{pmatrix} d\theta \right] \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}.$$

The two components correspond to the two transport directions, dr and $d\theta$. All in all, the connection has eight components and requires three indices: Γ^i_{jm}. Our job is to find these components! (In departure from the usual tensor-index notation, we wrote all vector components with downstairs indices, which avoids extra parentheses and confusion about upstairs indices and exponents.)

To find the connection coefficients, we consider four special cases (see the diagram):

- A small vector pointing in the *radial* direction, $\vec{v} = (v_r, 0)^T$, being parallel transported in the *radial* direction by dr:
  → The components of the parallel-transported vector remain unchanged.
- A small vector pointing in the *angular* direction, $\vec{v} = (0, v_\theta)^T$, being parallel transported in the *radial* direction by dr:
  → Because the length of the transported vector stays the same, $(r + dr)v'_\theta = rv_\theta$, its angular component shrinks to $v'_\theta = r/(r + dr)\ v_\theta \approx (r - dr)/r\ v_\theta = v_\theta - 1/r\ v_\theta dr$.
- A small vector pointing in the *radial* direction, $\vec{v} = (v_r, 0)^T$, being parallel transported in the *angular* direction by $d\theta$:
  → The transported vector acquires a negative angular component such that $rv'_\theta = -v_r d\theta$. Thus, $v'_\theta = -1/r\ v_r d\theta$ or, equivalently, $v'_\theta = v_\theta - 1/r\ v_r d\theta$.
- A small vector pointing in the *angular* direction, $\vec{v} = (0, v_\theta)^T$, being parallel transported in the *angular* direction by $d\theta$:
  → The transported vector acquires a positive radial component: $v'_r = |\vec{v}|d\theta = rv_\theta d\theta$ or, equivalently, $v'_r = v_r + rv_\theta d\theta$.

Linearly combining these special cases yields the general case:

$$\begin{pmatrix} v'_r \\ v'_\theta \end{pmatrix} = \begin{pmatrix} v_r \\ v_\theta \end{pmatrix} - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix} dr + \begin{pmatrix} 0 & -r \\ 1/r & 0 \end{pmatrix} d\theta \right] \begin{pmatrix} v_r \\ v_\theta \end{pmatrix}.$$

Thus, the components of the connection $\Gamma(r, \theta)$ are

$$\begin{pmatrix} \begin{pmatrix} \Gamma^r_{rr} & \Gamma^r_{\theta r} \\ \Gamma^\theta_{rr} & \Gamma^\theta_{\theta r} \end{pmatrix} \\ \begin{pmatrix} \Gamma^r_{r\theta} & \Gamma^r_{\theta\theta} \\ \Gamma^\theta_{r\theta} & \Gamma^\theta_{\theta\theta} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix} \\ \begin{pmatrix} 0 & -r \\ 1/r & 0 \end{pmatrix} \end{pmatrix}.$$

The connection plays an important role in defining the *covariant derivative*, which replaces the ordinary derivative when curved coordinates are used. Specifically, the vector derivatives $\partial v_i/\partial m$ must be upgraded to the covariant derivatives $\partial v_i/\partial m + \sum_j \Gamma^i_{jm} v_j$, where the second part corrects for the curviness of the coordinates. For the case of polar coordinates, we need to upgrade as follows (no upgrade needed for $\partial v_r/\partial r$):

$$\frac{\partial v_\theta}{\partial r} \to \frac{\partial v_\theta}{\partial r} + \frac{1}{r} v_\theta, \qquad \frac{\partial v_r}{\partial \theta} \to \frac{\partial v_r}{\partial \theta} - rv_\theta, \qquad \frac{\partial v_\theta}{\partial \theta} \to \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r} v_r.$$