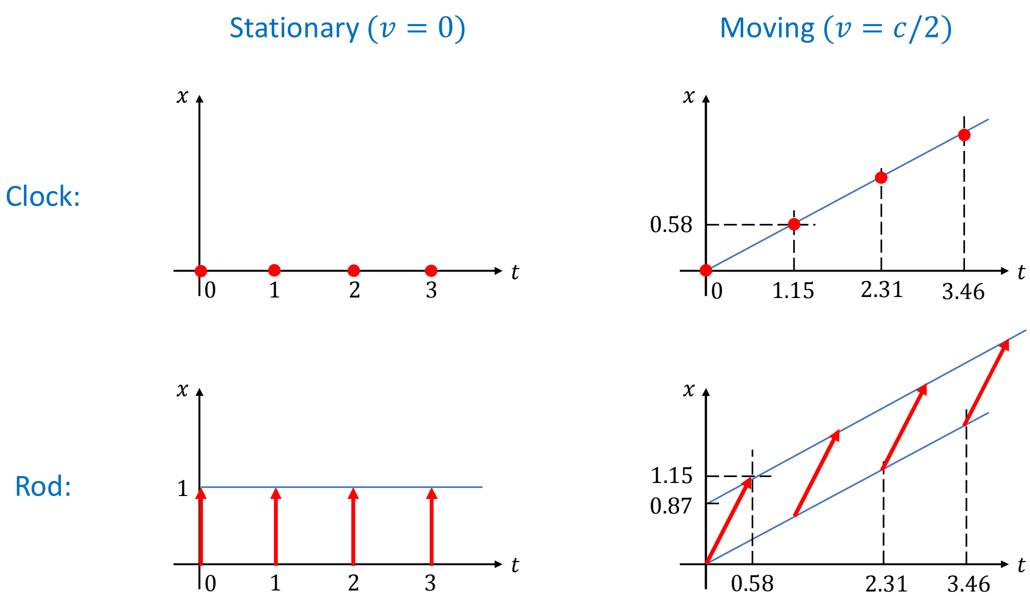
9.13 Time Dilation, Lorentz Contraction, and Rest Energy



Clocks that move at high speed appear to slow down to a stationary observer. This effect is known as *time dilation.* Moreover, rods that move at high speed appear to shorten in the direction of motion to a stationary observer. This effect is known as *Lorentz contraction*. In the following, we want to use the Lorentz transformation to understand these effects!

To be specific, we use the Lorentz transformation that boosts events (= space-time vectors) in the x-direction to half the speed of light, $v = c/2$ [QFTGA, Ch. 9.5]:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \approx \begin{pmatrix} 1.15 & 0.58 \\ 0.58 & 1.15 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix},$$

$y' = y$, and $z' = z$, where the rapidity is $\phi = \tanh^{-1}(v/c) = \tanh^{-1}(0.5) \approx 0.55$. In these formulas, the space-time coordinates x, y, z, and t are measured in units for which $c = 1$.

To get a handle on space transformations, such as 3D rotations, it is a good idea to study how the basis vectors transform. Let's try this strategy for the above space-time transformation! The basis vector $(1, 0)^T$ transforms to $(1.15, 0.58)^T$, that is, its time component gets larger: this looks like time dilation. But what does it actually mean to say that a vector points in the direction of time? Next, the basis vector $(0, 1)^T$ transforms to $(0.58, 1.15)^T$, that is, its space component gets larger as well. But shouldn't we get a *contraction*?

To make sense of this, we need to think in terms of *world lines*, that is, sets of events associated with a particular object, rather than individual space-time vectors, as we did above! The top-left diagram shows the world line of a stationary ticking clock. The horizontal line at $x = 0$ is the world line and the dots at $t = 0, 1, 2, 3, \ldots$ symbolize the ticks. The bottom-left diagram shows a stationary rod of length one. Every point on the rod has its own world line, but for simplicity only the ones at the top and the bottom are shown. All the world lines together fill the area of the horizontal band between $x = 0$ and $x = 1$, which is known as a *world sheet*. The arrows show the rod at $t = 0, 1, 2, 3, \ldots$

Time Dilation. Now, let's boost the clock to half the speed of light! To do that, we Lorentz transform the individual tick events: $(0, 0)^T$ goes to $(0, 0)^T$, $(1, 0)^T$ goes to $(1.15, 0.58)^T$, $(2, 0)^T$ goes to $(2.31, 1.15)^T$, $(3, 0)^T$ goes to $(3.46, 1.73)^T$, etc., as shown in the top-right diagram. The transformed ticks are located on a straight line with slope ½, which is the transformed worldline of the clock. A stationary observer sees the ticks spaced apart by about 15% more than those of a clock at rest. That's time dilation!

Time dilation can be grasped intuitively by considering a light clock consisting of a top and bottom mirror spaced apart by $c \cdot 1\text{sec} \approx 300{,}00\text{km}$ and a light pulse bouncing up and down between these two mirrors. If the light clock is stationary, it takes 1 second for the pulse to travel from one mirror to the other. If the light clock moves at the speed v, the light pulse has to travel a longer distance, namely $\sqrt{(c \cdot 1\text{sec})^2 + (vt)^2}$. Because the speed of light is independent of the inertial frame, the stationary observer sees the light pulse traveling for $t = \sqrt{(1\text{sec})^2 + (vt/c)^2}$. Solving for t yields $t = 1\text{sec}/\sqrt{1 - (v/c)^2}$. For $v/c = 0.5$ this evaluates to 1.15sec, confirming the 15% time dilation.

Lorentz Contraction. Next, we boost the rod to half the speed of light by Lorentz transforming its end-point events. The lower end-point events (at one second intervals) transform in the same way as the clock ticks discussed above. The upper end-point events transform as follows: $(0, 1)^T$ goes to $(0.58, 1.15)^T$, $(1, 1)^T$ goes to $(1.73, 1.73)^T$, $(2, 1)^T$ goes to $(2.89, 2.31)^T$, $(3, 1)^T$ goes to $(4.04, 2.89)^T$, etc., as shown in the bottom-right diagram. The transformed events mark the boundary of the transformed world sheet of the rod. A stationary observer measures the length of the rod at any given time (equal-time length) as 13% shorter (0.87×) than its rest length. That's Lorentz contraction!

But there is more to this story: Although moving objects are *measured* to be contracted, they *appear* to be rotated! The difference is that a measurement compares points in space at the same time, as explained above, whereas an appearance refers to photons entering our eyes at the same time. Since the photons from the rear of the object need to travel longer to reach our eyes than those from the front, there is a difference. In particular, if the moving object is a ball, it always appears as a ball (a rotated ball is still a ball) and not as an ellipsoid, regardless of the speed it is moving at [RtR, Ch. 18.5]. This effect is known as *Terrell-Penrose rotation*.

Rest Energy and Energy Dilation. A Lorentz transformation acts on energy just like it acts on time. In fact, energy and time are closely related. For example, Noether's theorem associates energy conservation with time-translation symmetry and the quantum-mechanical operator for the energy observable is $i\hbar\partial/\partial t$. Just like a time interval undergoes time dilation, so does an object with rest energy E_0 undergo "energy dilation" to $E = E_0/\sqrt{1 - (v/c)^2} \approx E_0 + \frac{1}{2}E_0(v/c)^2$ when moving at speed v. From Newtonian mechanics we know what the dilation amount is: it is the kinetic energy $\frac{1}{2}mv^2$. By equating $\frac{1}{2}E_0(v/c)^2 = \frac{1}{2}mv^2$, we discover that the rest energy must be $E_0 = mc^2$.

Note: in the above diagrams, we drew the time axis from left to right, a direction considered natural by many people; however, physicists prefer to draw the time axis from bottom to top.