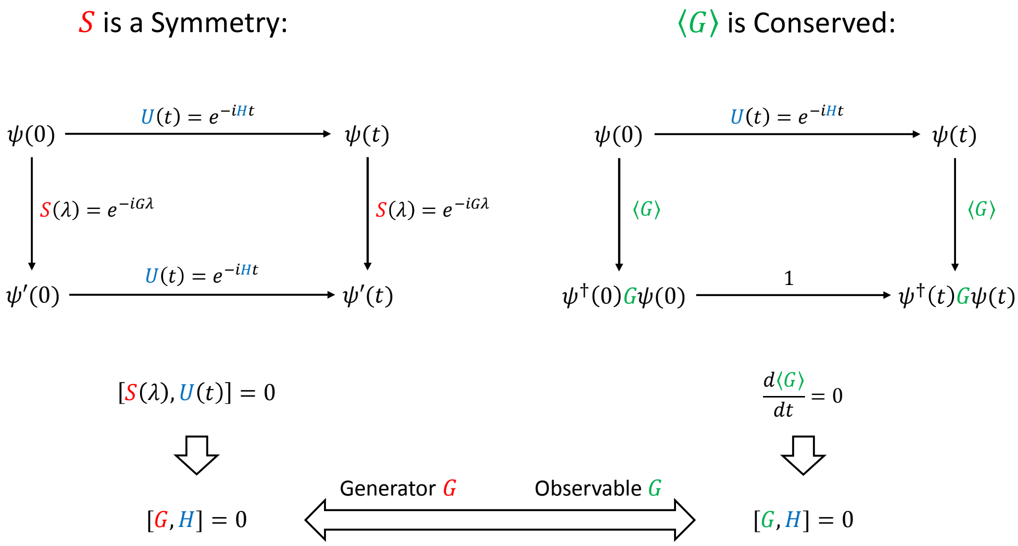
9.3 Symmetry and Conservation in Quantum Mechanics



In quantum mechanics, the state of a system is given by the complex vector ψ, which is an element of *Hilbert space*. (More accurately, a physical state is given by an equivalence class of vectors related by phase factors.) This state evolves according to *Schrödinger's equation* $i\partial\psi/dt = H\psi$, where H is the *Hamiltonian*, a Hermitian operator (matrix) acting on Hilbert space, and we chose units for which $\hbar = 1$. Integrating this differential equation yields the explicit time evolution $\psi(t) = U(t)\psi(0)$, where $U(t) = e^{-iHt}$ is a unitary operator and we assumed that H is time independent. The matrix exponential is to be interpreted as the power series $e^{-iHt} = I - iHt + \frac{1}{2}(-iHt)^2 + \cdots$.

What is a symmetry transformation? Let's define the time-independent unitary operator $S(\lambda)$ that transforms the original state to the primed state like $\psi' = S(\lambda)\psi$, where λ is the transformation parameter. Now, if the transformed state evolves exactly in sync to the original state, meaning $\psi'(t) = S(\lambda)\psi(t)$ for all times t, then $S(\lambda)$ is a symmetry transformation. Alternatively, we can say that if evolving the initial state first and transforming it second, $S(\lambda)U(t)\psi$, results in the same final state as transforming it first and evolving it second, $U(t)S(\lambda)\psi$, for all times t, then $S(\lambda)$ is a symmetry transformation. In other words, a symmetry transformation leaves the law of time evolution unaffected. Mathematically, we have $U(t)S(\lambda) = S(\lambda)U(t)$ or, equivalently, $[S(\lambda), U(t)] = 0$ for all t, where $[\cdot\,,\,\cdot]$ is the commutator bracket.

Summary: a symmetry transformation commutes with the time evolution (left side of the diagram).

Making use of $U(t) = e^{-iHt}$, the condition $[S(\lambda), U(t)] = 0$ for all t can also be written as $[S(\lambda), H] = 0$. This can be demonstrated by expanding the matrix exponential into its power series. Furthermore, the unitary transformation can be written as $S(\lambda) = e^{-iG\lambda}$, where G is the generator of the transformation (times i). Now, the condition $[S(\lambda), H] = 0$ can be rewritten as $[G, H] = 0$. Again, this can be demonstrated by expanding the matrix exponential into its power series $e^{-iG\lambda} = I - iG\lambda + \frac{1}{2}(-iG\lambda)^2 + \cdots$.

Summary: the generator of a symmetry transformation commutes with the Hamiltonian (left side of the diagram).

What is a conserved quantity? A quantum-mechanical observable is represented by an operator, G. The expectation value of this observable is given by $\langle G \rangle = \psi^\dagger G\psi$. Knowing how ψ evolves in time, we can calculate how this expectation value evolves in time (from time 0 to time t): $\langle G(t) \rangle = \psi(t)^\dagger G\psi(t) = [U(t)\psi(0)]^\dagger G[U(t)\psi(0)] = \psi(0)^\dagger U^{-1}(t)GU(t)\psi(0)$. For this value to be conserved (= to be time independent), we need $U^{-1}(t)GU(t) = G$ or, equivalently, $[G, U(t)] = 0$ for all t. It can be shown that this condition not only conserves the expectation value but the entire probability distribution of the observable. Finally, making use of $U(t) = e^{-iHt}$, the condition $[G, U(t)] = 0$ for all t can be rewritten as $[G, H] = 0$.

Summary: a conserved observable commutes with the Hamiltonian (right side of the diagram).

How are symmetry and conservation related? From the above arguments, we conclude: if $S(\lambda)$ is a symmetry transformation, then the associated generator G is a conserved observable, meaning that the expectation value $\langle G(t) \rangle = \psi(t)^\dagger G\psi(t)$ and the probability distribution of the observable are time independent. Conversely, if G is a conserved observable, then taking the exponential map yields a symmetry transformation, $S(\lambda) = e^{-iG\lambda}$.

To understand this better, let's assume that, in addition to $[G, H] = 0$, the initial state $\psi(0)$ is an *eigenstate* of the conserved observable G, that is, $\psi(0) = \phi_i$ for some i, where $G\phi_i = g_i\phi_i$. Because $[G, H] = 0$, this state is also an eigenstate of the Hamiltonian H, that is, $H\phi_i = E_i\phi_i$. For this special state the observable G has a definite value, namely g_i, and the energy has a definite value, namely E_i. Both values are time independent (= conserved). In this case, the symmetry transformation $S(\lambda) = e^{-iG\lambda}$ simply multiplies the state with a phase factor, namely $\psi'(0) = e^{-iG\lambda}\psi(0) = e^{-ig_i\lambda}\psi(0)$, and likewise the time evolution $U(t) = e^{-iHt}$ multiplies the state with another phase factor, namely $\psi(t) = e^{-iHt}\psi(0) = e^{-iE_it}\psi(0)$. Clearly, the two phase factors commute. Note that the symmetry transformation changes the phase of the state by $\Delta\varphi = g_i\Delta\lambda$ and therefore g_i is the *rate of change* of the phase with respect to λ. In other words, for these definite-value states, we can interpret the value of the conserved observable as the rate of change of the quantum-mechanical phase with respect to the parameter of the associated symmetry transformation [FLP, Vol. III, Ch. 17].

For example, the rate of change of the quantum-mechanical phase with respect to the angle of rotation about the z axis is the angular momentum about the z axis (assuming the state has a definite angular momentum about the z axis). Similarly, the rate of change of the phase with respect to the amount of translation along the x axis is the momentum in the x direction (assuming the state has a definite momentum in the x direction). Note that phase change per distance is a wavenumber and thus has the right dimension for momentum ($\hbar = 1$). Finally, the rate of change of the phase with respect to the amount of time translation is the energy (assuming the state has a definite energy). Note that phase change per time interval is an (angular) frequency and thus has the right dimension for energy.