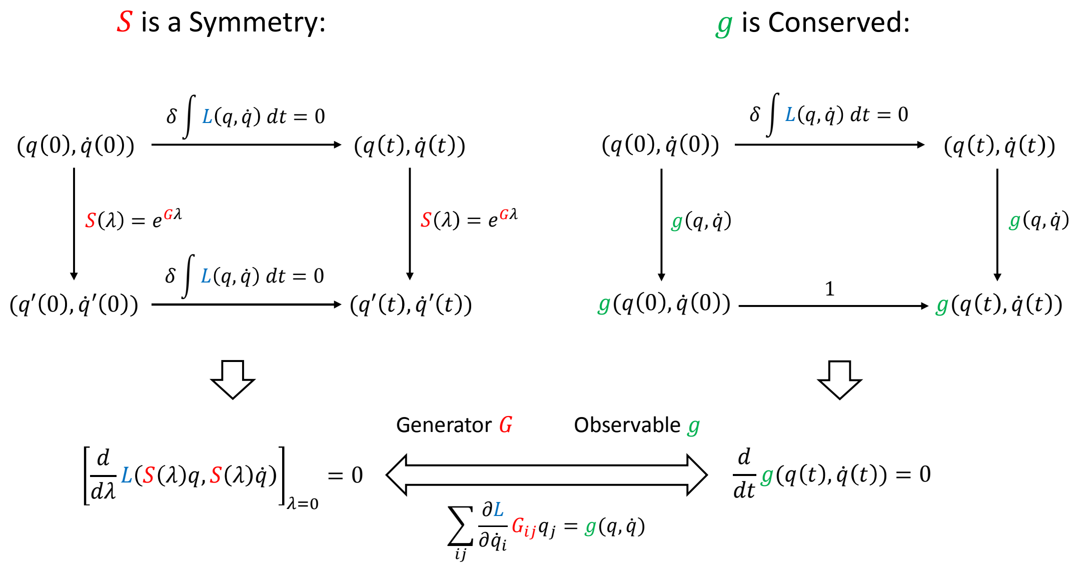
9.4 Symmetry and Conservation in Classical Lagrangian Mechanics



In the (classical) Lagrangian framework, a system is described in terms of the (generalized) position coordinates q_i (= degrees of freedom), which define a point in *configuration space*. The state of the system is given by these coordinates and their time derivatives $\dot q_i$, which together define a point $(q,\dot q)$ in *velocity phase space* (= tangent bundle of configuration space). This state evolves in time from t_1 to t_2 such that the *action* of the system becomes stationary: $\delta \int_{t_1}^{t_2} L(q,\dot q)\, dt = 0$, where $L(q,\dot q)$ is the *Lagrangian* (written without an explicit time dependence of L, which implies energy conservation). The explicit time evolution is found by varying the trajectory in configuration space while keeping the endpoints fixed. The result can be written as $q_i(t) = T(t,q)q_i(0)$ and $\dot q_i(t) = T(t,q,\dot q)\dot q_i(0)$, where $T(t,q,\dot q)$ is a real operator acting on velocity phase space. For more information on Lagrangian mechanics, see [TM, Vol. 1, Ch. 6-7; NNCM, Ch. 4; RtR, Ch. 20.1].

What is a symmetry transformation? Let's define the time-independent *point transformation* $S(\lambda,q,\dot q)$ that maps the original state to the primed state according to $q_i' = S(\lambda,q)q_i$ and $\dot q_i' = S(\lambda,q,\dot q)\dot q_i$, where λ is the transformation parameter (S is not necessarily linear in q). Now, if evolving the initial state first and transforming it second, $S(\lambda)T(t)(q(0),\dot q(0))$, results in the same final state as transforming it first and evolving it second, $T(t)S(\lambda)(q(0),\dot q(0))$, for all times t, then $S(\lambda)$ is a symmetry transformation. Thus, just like in quantum mechanics, a symmetry transformation commutes with the time evolution: $[S(\lambda),T(t)] = 0$ (see the left side of the diagram). From a slightly different viewpoint, we can say that $S(\lambda)$ is a symmetry transformation if the action $\int_{t_1}^{t_2} L(S(\lambda)q_i, S(\lambda)\dot q_i))\, dt$ does not depend on λ, in other words, if the action, which encodes the law of time evolution, is not affected by the symmetry transformation.

In the following, we restrict ourselves to *linear* point transformations, that is, we take $S(\lambda,q)$ to be a matrix that acts on the state vector according to $q_i' = S_{ij}(\lambda)q_j$ and $\dot q_i' = S_{ij}(\lambda)\dot q_j$ (Einstein summation convention implied), where $S_{ij}(\lambda) = e^{G_{ij}\lambda}$ and G_{ij} is the *generator* of the transformation. Such transformations include rotations but not translations. Now, $S_{ij}(\lambda)$ is a symmetry transformation if the action $\int_{t_1}^{t_2} L(S_{ij}(\lambda)q_j, S_{ij}(\lambda)\dot{q}_j))\, dt$ does not depend on λ. Since large transformations can be built up from small ones, it is sufficient that the action is not affected by *small* transformations (i.e., transformations close to the identity): $[d/d\lambda \int_{t_1}^{t_2} L(S_{ij}(\lambda)q_j, S_{ij}(\lambda)\dot{q}_j))\, dt]_{\lambda=0} = 0$. In turn, for this to be true, it is sufficient that the Lagrangian (rather than the action) remains unchanged under small transformation. (While this condition is sufficient, it is not strictly necessary and can be relaxed [NNCM, Ch 10.4.1].) Thus, $S_{ij}(\lambda)$ is a symmetry transformation, if (see the left side of the diagram)

$$\left[\frac{d}{d\lambda} L\big(S_{ij}(\lambda)q_j, S_{ij}(\lambda)\dot{q}_j\big)\right]_{\lambda=0} = 0.$$

What is a conserved quantity? A classical observable is given by a real function of the state, $g(q, \dot{q})$. For this observable to be conserved, its total time derivative must vanish (see the right side of the diagram):

$$\frac{d}{dt} g\big(q(t), \dot{q}(t)\big) = 0.$$

How are symmetry and conservation related? We start by applying the chain rule to the above symmetry condition and use that the generator of the transformation is given by $G_{ij} = [dS_{ij}/d\lambda]_{\lambda=0}$

$$\frac{\partial L}{\partial q_i} G_{ij} q_j + \frac{\partial L}{\partial \dot{q}_i} G_{ij} \dot{q}_j = 0.$$

Next, we make use of the fact that the action is stationary for a valid solution $q_j(t)$, that is, it must satisfy the Euler-Lagrange equation. We replace $\partial L/\partial q_i$ on the left by $d/dt\ (\partial L/\partial \dot{q}_i)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) G_{ij} q_j + \frac{\partial L}{\partial \dot{q}_i} G_{ij} \dot{q}_j = 0.$$

Finally, we use the product rule to combine the two terms into a single time derivate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} G_{ij} q_j\right) = 0.$$

We have thus found the conserved quantity for a given symmetry generator G (see the diagram):

$$g(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}_i} G_{ij} q_j.$$

Introducing the canonical momentum vector $p_i = \partial L/\partial \dot{q}_i$ and using vector-matrix notation, we can write this conserved quantity as $g(t) = p(t)^T G q(t)$. This is Noether's theorem for particle theories (as opposed to field theories) restricted to linear point transformations. For more general versions of this theorem, see the Appendices "Noether's Theorem for Particle Theories", etc.

Whereas the relationship between symmetry and conservation in classical Lagrangian mechanics is less direct than in quantum mechanics, the results look very similar. In both cases, the generator of the symmetry plays a central role: $g(t) = p(t)^T G q(t)$ vs. $\langle G(t) \rangle = \psi(t)^\dagger G \psi(t)$. Classical mechanics comes closest in spirit to quantum mechanics when starting from an action principle: the Lagrangian formulation of classical mechanics exhibits a quantum-mechanical "imprint".