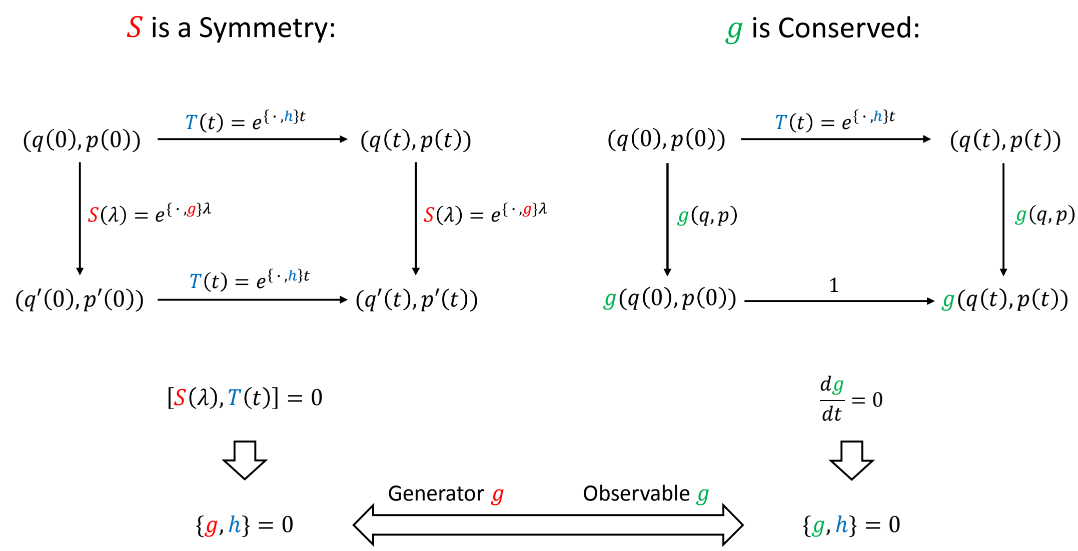
## 9.5 Symmetry and Conservation in Classical Hamiltonian Mechanics




In the (classical) Hamiltonian framework, the state of a system is given by the (generalized) position coordinates $q_i$ and the canonical momenta $p_i$ (conjugate to $q_i$), which together define a point $(q, p)$ in *(momentum) phase space* (= cotangent bundle of configuration space). This state evolves according to *Hamilton's equations* $\dot{q}_i = \partial h(q,p)/\partial p_i$ and $\dot{p}_i = -\partial h(q,p)/\partial q_i$, where $h(q,p)$ is the *Hamiltonian*, a function of phase space. Introducing the Poisson-bracket notation $\{f, g\} \coloneqq \sum_i (\partial f/\partial q_i \cdot \partial g/\partial p_i - \partial f/\partial p_i \cdot \partial g/\partial q_i)$, we can rewrite these equations more systematically as $\dot{q}_i = \{q_i, h\}$ and $\dot{p}_i = \{p_i, h\}$. Integrating these differential equations yields the explicit time evolution $q_i(t) = T(t,q,p)q_i(0)$ and $p_i(t) = T(t,q,p)p_i(0)$, where $T(t,q,p) = e^{\{\cdot,h\}t}$ assuming that $h(q,p)$ is time independent. The exponential expression $q_i(t) = e^{\{\cdot,h\}t}q_i(0)$ is to be interpreted as the power series $q_i(t) = [q_i + \{q_i, h\}t + \frac{1}{2}\{\{q_i, h\}, h\}t^2 + \cdots]_{q_i = q_i(0)}$ and the same comment applies to $p_i(t)$. For more information on Hamiltonian mechanics, see [TM, Vol. 1, Ch. 8-10; NNCM, Ch. 5; RtR, Ch. 20.2].

**What is a symmetry transformation?** A time-independent transformation acting on phase space can be given by specifying the generator function $g(q,p)$ to be used with the differential equations $dq_i/d\lambda = \{q_i, g(q,p)\}$ and $dp_i/d\lambda = \{p_i, g(q,p)\}$, where $\lambda$ is the transformation parameter. Integrating these equations yields the explicit *canonical transformation* $q_i' = S(\lambda,q,p)q_i$ and $p_i' = S(\lambda,q,p)p_i$, where $S(\lambda,q,p) = e^{\{\cdot,g\}\lambda}$ [NNCM, Ch. 7.3.3]. Now, if evolving the initial state first and transforming it second, $S(\lambda)T(t)(q(0),p(0))$, results in the same final state as transforming it first and evolving it second, $T(t)S(\lambda)(q(0),p(0))$, for all times $t$, then $S(\lambda)$ is a symmetry transformation. Mathematically, we have $[S(\lambda), T(t)] = [e^{\{\cdot,g\}\lambda}, e^{\{\cdot,h\}t}] = 0$ for all $t$.

Summary: a symmetry transformation commutes with the time evolution (left side of the diagram).

Let's express the above condition in terms of the generator functions $g(q,p)$ and $h(q,p)$. Expanding the exponentials into power series, we find that the exponentials commute if $\{\{\cdot, g\}, h\} = \{\{\cdot, h\}, g\}$ or, more explicitly, if $\{\{f, g\}, h\} = \{\{f, h\}, g\}$ holds for any function $f(q,p)$. Using the Jacobi identity,

$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0$, we can see that this is equivalent to the condition $\{g,h\}=0$. (While this symmetry condition is sufficient, it is not strictly necessary and could be relaxed [NNCM, Ch. 10.3.1].) We can interpret the condition $\{h,g\}=0$ (which is identical to $\{g,h\}=0$) as saying that the symmetry generator $g$ leaves the Hamiltonian, which encodes the law of time evolution, invariant.

Summary: the generator of a symmetry transformation "Poisson commutes" with the Hamiltonian (left side of the diagram).

How is the generator function $g(q,p)$ in this example (Hamiltonian formalism) related to the generator matrix $G$ in the previous example (Lagrangian formalism)? To find out, we compare the transformation $q_i' = e^{\{\cdot,g\}\lambda} q_i$ with the transformation $q_i' = e^{G_{ij}\lambda} q_j$. Taking the derivative with respect to $\lambda$ and evaluating the result at the identity, the *transformation rates* are $\partial q_i'/\partial\lambda = \{q_i, g(p,q)\}$ and $\partial q_i'/\partial\lambda = G_{ij}q_j$, respectively. These rates agree, if $\{q_i, g(q,p)\} = G_{ij}q_j$ or, after expanding the Poisson bracket, $\partial g(q,p)/\partial p_i = G_{ij}q_j$. After integration with respect to $p_i$, we find $g(q,p) = p_i G_{ij} q_j$ or, in vector-matrix notation, $g(q,p) = p^T G q$. Thus, given the generator matrix $G$ of a point transformation, the corresponding generator function in the Hamiltonian formalism is $p^T G q$. (Incidentally, if the position vector transform like $q_i' = e^{G_{ij}\lambda} q_j$, then, to be consistent with the canonical transformation generated by the function $p^T G q$, the momentum vector must transform like $p_i' = e^{-G_{ji}\lambda} p_j$. Thus, in the case of orthogonal transformations, where $G_{ij} = -G_{ji}$, the position and momentum coordinates transform identically.)

**What is a conserved quantity?** A classical observable is given by a real function of the state: $g(q,p)$. Its value at time zero is $g(q(0),p(0))$ and evolves to $g(q(t),p(t))$ at time $t$. Hamilton's equations $\dot{q}_i = \{q_i,h\}$ and $\dot{p}_i = \{p_i,h\}$ can be generalized for an arbitrary observable $g(q,p)$. Evaluating $\dot{g}(q,p)$ using the chain rule, we find

$$\dot{g}(q,p) = \sum\nolimits_i \left(\frac{\partial g}{\partial q_i}\{q_i,h\} + \frac{\partial g}{\partial p_i}\{p_i,h\}\right) = \sum\nolimits_i \left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right).$$

Rewriting this with a Poisson bracket yields the generalized Hamilton equation $\dot{g} = \{g,h\}$. It follows that for $g$ to be conserved (= to be time independent), we need $\{g,h\}=0$.

Summary: the phase-space function of a conserved observable "Poisson commutes" with the Hamiltonian (right side of the diagram).

**How are symmetry and conservation related?** From the above arguments, we conclude: if the function $g(q,p)$ generates a symmetry transformation, then this same function is a conserved observable. Conversely, if $g(q,p)$ is a conserved observable, then this same function generates a symmetry transformation.

The relationship between symmetry and conservation in the Hamiltonian formulation of classical mechanics closely parallels that of quantum mechanics. The Poisson brackets in classical Hamiltonian mechanics correspond to the commutator brackets in quantum mechanics.