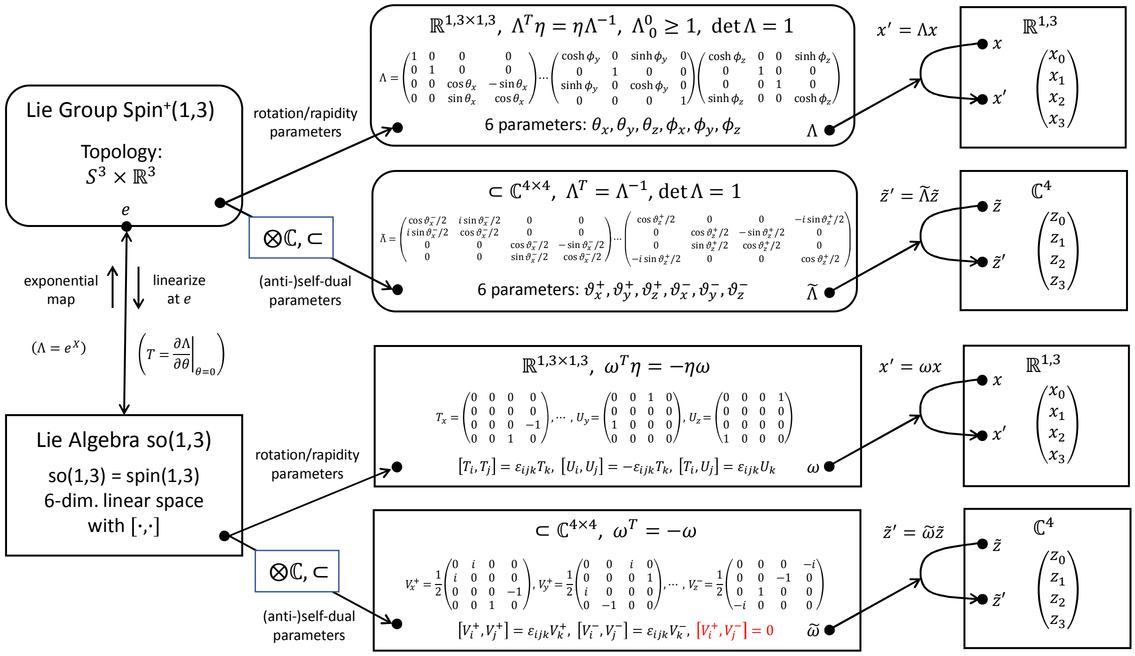
5.21 Spin⁺(1,3): The Double Cover of SO⁺(1,3); so(1,3)$_\mathbb{C}$ = so(3)$_\mathbb{C}$ ⊕ so(3)$_\mathbb{C}$



When discussing SO(4), we were able to split the group and its algebra into two independent parts by changing the parameters from θ_k and ϕ_k to the linear combinations $\vartheta_k^+ = \theta_k + \phi_k$ and $\vartheta_k^- = \theta_k - \phi_k$. Can we do something similar for SO⁺(1,3)?

Inspired by our experience with the self-dual and anti-self-dual representations, we may try the parameter combinations $\vartheta_k^+ = \theta_k + i\phi_k$ and $\vartheta_k^- = \theta_k - i\phi_k$. Substituting $\theta_k = \frac{1}{2}(\vartheta_k^+ + \vartheta_k^-)$ and $\phi_k = \frac{1}{2}i(\vartheta_k^- - \vartheta_k^+)$ into the original Lorentz transformation (shown in the upper branch of the diagram), splitting each factor into a matrix that depends on ϑ_x^+ and one that depends on ϑ_x^-, and combining matrices that depend on the same parameter yields the new transformation $\tilde{\Lambda} = \Lambda_x^+(\vartheta_x^-) \cdot \Lambda_x^-(\vartheta_x^+) \cdot \Lambda_y^+(\vartheta_y^-) \cdot \Lambda_y^-(\vartheta_y^+) \cdot \Lambda_z^+(\vartheta_z^-) \cdot \Lambda_z^-(\vartheta_z^+)$, where

$$\Lambda_x^+ = \begin{pmatrix} \cos\vartheta_x^-/2 & i\sin\vartheta_x^-/2 & 0 & 0 \\ i\sin\vartheta_x^-/2 & \cos\vartheta_x^-/2 & 0 & 0 \\ 0 & 0 & \cos\vartheta_x^-/2 & -\sin\vartheta_x^-/2 \\ 0 & 0 & \sin\vartheta_x^-/2 & \cos\vartheta_x^-/2 \end{pmatrix}, \; \Lambda_x^- = \begin{pmatrix} \cos\vartheta_x^+/2 & -i\sin\vartheta_x^+/2 & 0 & 0 \\ -i\sin\vartheta_x^+/2 & \cos\vartheta_x^+/2 & 0 & 0 \\ 0 & 0 & \cos\vartheta_x^+/2 & -\sin\vartheta_x^+/2 \\ 0 & 0 & \sin\vartheta_x^+/2 & \cos\vartheta_x^+/2 \end{pmatrix},$$

$$\Lambda_y^+ = \begin{pmatrix} \cos\vartheta_y^-/2 & 0 & i\sin\vartheta_y^-/2 & 0 \\ 0 & \cos\vartheta_y^-/2 & 0 & \sin\vartheta_y^-/2 \\ i\sin\vartheta_y^-/2 & 0 & \cos\vartheta_y^-/2 & 0 \\ 0 & -\sin\vartheta_y^-/2 & 0 & \cos\vartheta_y^-/2 \end{pmatrix}, \; \Lambda_y^- = \begin{pmatrix} \cos\vartheta_y^+/2 & 0 & -i\sin\vartheta_y^+/2 & 0 \\ 0 & \cos\vartheta_y^+/2 & 0 & \sin\vartheta_y^+/2 \\ -i\sin\vartheta_y^+/2 & 0 & \cos\vartheta_y^+/2 & 0 \\ 0 & -\sin\vartheta_y^+/2 & 0 & \cos\vartheta_y^+/2 \end{pmatrix},$$

$$\Lambda_z^+ = \begin{pmatrix} \cos\vartheta_z^-/2 & 0 & 0 & i\sin\vartheta_z^-/2 \\ 0 & \cos\vartheta_z^-/2 & -\sin\vartheta_z^-/2 & 0 \\ 0 & \sin\vartheta_z^-/2 & \cos\vartheta_z^-/2 & 0 \\ i\sin\vartheta_z^-/2 & 0 & 0 & \cos\vartheta_z^-/2 \end{pmatrix}, \; \Lambda_z^- = \begin{pmatrix} \cos\vartheta_z^+/2 & 0 & 0 & -i\sin\vartheta_z^+/2 \\ 0 & \cos\vartheta_z^+/2 & -\sin\vartheta_z^+/2 & 0 \\ 0 & \sin\vartheta_z^+/2 & \cos\vartheta_z^+/2 & 0 \\ -i\sin\vartheta_z^+/2 & 0 & 0 & \cos\vartheta_z^+/2 \end{pmatrix}$$

(the superscripts of $\Lambda_i^\pm$ will make sense in a moment). Similar to what happened for SO(4), the new matrix $\tilde{\Lambda}$ is no longer a representation of SO⁺(1,3) but of its double cover, Spin⁺(1,3). In other words, each element of SO⁺(1,3) is now labeled by two distinct sets of parameter values. Moreover, for $\tilde{\Lambda}$ to remain real, as required for a Lorentz transformation, the new parameters must be restricted to be complex conjugates of each other: $\vartheta_k^- = (\vartheta_k^+)^*$. If, instead, we permit generally complex parameters ϑ_k^+ and ϑ_k^-, the set of transformations grows from depending on six to twelve real parameters and the transformations become complex. This larger group is known as the *complexification* of Spin⁺(1,3), written as $\mathrm{Spin}^+(1,3)_\mathbb{C}$ or $\mathrm{Spin}^+(1,3) \otimes \mathbb{C}$ (see the lower branch of the diagram).

Taking the derivatives of the transformation matrix $\widetilde{\Lambda}$ with respect to the new parameters and setting them to zero yields the following basis generators:

$$V_x^+ = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ V_y^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ V_z^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$V_x^- = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ V_y^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ V_z^- = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

These basis generators are linear combinations of the old ones $V_k^+ = \frac{1}{2}(T_k + iU_k)$ and $V_k^- = \frac{1}{2}(T_k - iU_k)$ (now the ± superscripts now make!). Exponentiating these generators and pairing them appropriately confirms that they reproduce the original Lorentz transformations: $\exp[\frac{1}{2}(T_k + iU_k)(\theta_k - i\phi_k)] \cdot \exp[\frac{1}{2}(T_k - iU_k)(\theta_k + i\phi_k)] = \exp(T_k\theta_k) \cdot \exp(U_k\phi_k)$ (note that $[T_k, U_k] = 0$). Evaluating the commutators shows that all three V_i^+ commute with all three V_j^-, that is, $[V_i^+, V_j^-] = 0$, and thus the algebra splits in two parts, as we were hoping for! Checking the remaining commutation relations reveals $[V_i^+, V_j^+] = \varepsilon_{ijk}V_k^+$ and $[V_i^-, V_j^-] = \varepsilon_{ijk}V_k^-$, that is, the same relations as for so(3). (For $J_k = iT_k$ and $K_k = iU_k$, we get the new generators $N_k^\pm = \frac{1}{2}(J_k \pm iK_k)$ with the commutation relations $[N_i^+, N_j^-] = 0$, $[N_i^+, N_j^+] = i\varepsilon_{ijk}N_k^+$, and $[N_i^-, N_j^-] = i\varepsilon_{ijk}N_k^-$ [PfS, Ch. 3.7.3].)

Does so(1,3) break up into so(3) ⊕ so(3)? No, not quite! The new basis generators V_k^+ and V_k^- do *not* span the same vector space over the reals as the old ones (T_k and U_k) and thus the new Lie algebra, which splits into two parts, is *not* so(1,3). However, the larger bases given by V_k^+, iV_k^+, V_k^-, iV_k^- and T_k, iT_k, U_k, iU_k do span the same vector space over the reals, which is now twelve dimensional. It is this *complexified Lie algebra* that splits into two parts: $\mathrm{so}(1,3)_\mathbb{C} = \mathrm{so}(3)_\mathbb{C} \oplus \mathrm{so}(3)_\mathbb{C}$. See the Appendix "From Rotation to Lorentz Transformation" for more information.

We can think of Spin⁺(1,3) as a 6-dimensional subgroup, a so-called *real form*, of the 12-dimensional group $\mathrm{Spin}^+(1,3)_\mathbb{C}$. The complexified group $\mathrm{Spin}^+(1,3)_\mathbb{C}$ is isomorphic to $\mathrm{Spin}(4)_\mathbb{C}$ and also has Spin(4) and Spin⁺(2,2) as possible real forms. Whereas the latter two real forms split into two parts, Spin⁺(1,3) does *not* [QTGR, Ch. 40.4].

Without the ability to cleanly split so(1,3) into so(3) ⊕ so(3), how can we systematically enumerate the irreducible representations of so(1,3)? It turns out that if we limit ourselves to the finite-dimensional irreducible representations, we can still enumerate them by specifying two representations of so(3). This can be shown with *Weyl's unitarian trick* [Wikipedia: Representation theory of the Lorentz group]. In other words, the naming scheme (j_1, j_2) that we introduced for the representations of so(4) carries over to so(1,3)! See the Appendix "The Irreducible Representations of the Lorentz Group".