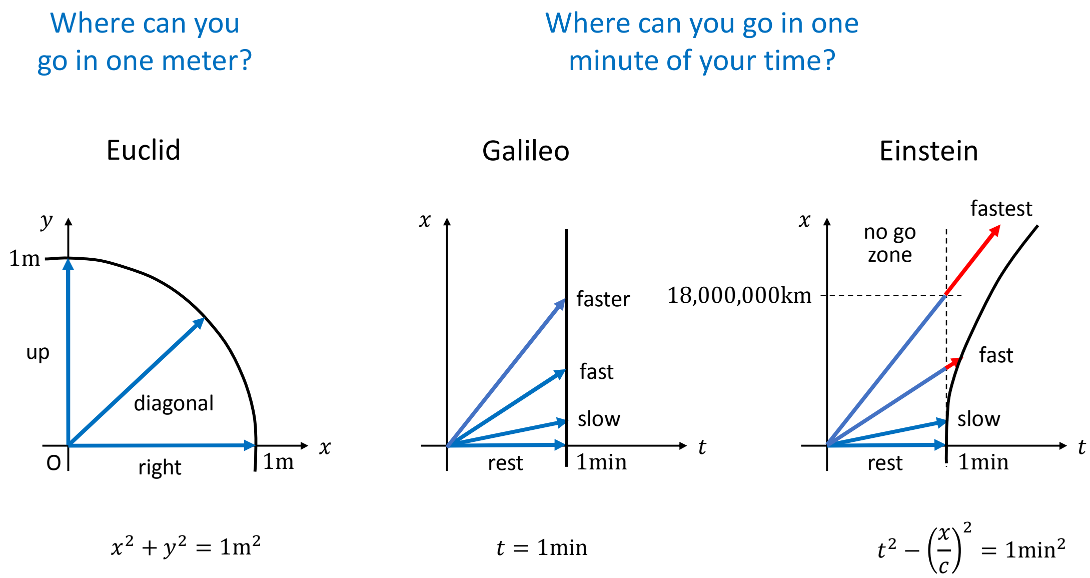
9.12 Euclid, Galileo, Einstein: Distance in Space and Time



To sharpen our intuition about the "distance" (= interval) between two points in space-time (= two events) according to the special theory of relativity, we compare three situations: (i) distance in Euclidean space, (ii) "distance" in pre-relativistic Galilean space-time, and (iii) "distance" in space-time according to Einstein. Comparing these three distance measures will help us to understand *time dilation* and the fact that we can travel arbitrarily far in a finite amount of (proper) time despite being limited by the speed of light.

Euclid. Let's say that we are located at point O in space and allowed to move by one meter, where can we go? That's easy: all possible destinations are located on a sphere with center O and radius one meter, or, if we restrict ourselves to two dimensions, on a *circle* with radius one meter. The diagram on the left-hand side illustrates the 2-dimensional case. We can go to the right or up or in any other direction, but we always end up at a point on the circle. When starting from the origin and using Cartesian coordinates, we know, thanks to Pythagoras and Euclid, that all possible destinations are given by the x and y coordinates that satisfy $x^2 + y^2 = 1\text{m}^2$.

Galileo. Next, let's bring time into the mix. We are again at the origin of the coordinate system and the time is zero. Where can we go in one minute according to our watch? We can go to any space-time point that has a time coordinate equal to one minute, or so Galileo and Newton thought. The diagram in the center illustrates this situation for a single space dimension (x coordinate) and the time coordinate t: we can stand still, go slowly, or go fast, but we always end up at a point on the vertical line with $t = 1\text{min}$. To reach point x in space, we need to move at the speed $v = x/1\text{min}$. Different directions of the arrow in this diagram correspond to different speeds: an arrow to the right means we are standing still, straight up means we are moving infinitely fast, and the directions in between correspond to intermediate speeds.

Einstein. The space-time diagram that we just discussed is not correct, at least it is not accurate at high speeds. Einstein's special theory of relativity asserts that in one minute according to our watch, which is known as *proper time*, we can go to any space-time point that lies on a particular hyperboloid. In the case of a single space dimension, the possible space-time destinations are on the *hyperbola* given by the x and t coordinates that satisfy $t^2 - (x/c)^2 = 1\text{min}^2$, where c is the speed of light, not the vertical line at $t = 1\text{min}$ that Galileo considered. The diagram on the right-hand side shows the correct situation: again, we can stand still, go slowly, or go fast, but after one minute of our time we always end up at a point on the hyperbola.

Let's compare the space-time diagram according to Galileo (center) to that according to Einstein (right):

- When moving slowly, say below 1% of the speed of light (< 3,000 km/s), the possible space-time destinations are close to the t axis where Einstein's hyperbola looks almost exactly like Galileo's vertical line. At low speeds, there is no significant difference between coordinate time t, relevant to a stationary observer, and proper time $\sqrt{t^2 - (x/c)^2}$, relevant to the moving observer.
- When moving fast, say above 50% of the speed of light (> 150,000 km/s), the bending of Einstein's hyperbola becomes noticeable and we can reach space-time points with a time coordinate larger than one minute (red arrow)! Specifically, the condition is $t^2 - (x/c)^2 = 1\text{min}^2$ or $t^2 - (vt/c)^2 = 1\text{min}^2$, which yields $t = 1\text{min}/\sqrt{1 - (v/c)^2}$ when solved for t. This means that our (proper) time must have slowed down relative to the stationary (coordinate) time, an effect known as *time dilation*. The faster we move, the more significant is the deviation of the hyperbola from a vertical line becomes. When we approach the speed of light, we can reach the farthest point on the hyperbola at $x \to \infty$. Interestingly, although our maximum speed is limited, we can reach *any* point in space (any x coordinate) in one minute of our time! Time dilation makes it possible.
- When moving at the speed of light (≈ 300,000 km/s or 18,000,000 km/min), time dilation becomes infinite. We can now go anywhere in no time at all! (Unfortunately, to move at the speed of light, we first have to become massless.) The space-time distance (proper time) between any two events on the same light ray is zero. The set of all events that are separated from the origin by zero space-time distance form two diagonal lines in our 2D diagram (only the line with positive x coordinates, labeled "fastest", is shown). If we plot two space dimensions instead of one, this set of events looks like two 3D cones, known as *light cones* or *null cones*.
- In the diagram according to Galileo, we can pass through any space-time point located between the origin and the vertical line. However, in the diagram according to Einstein, there are two triangular regions, one above and one below the origin, that we *cannot* pass through. The space-time points in these regions are said to be *space-like separated* from the origin. In fact, we can only move on *time-like* trajectories, which pass through the triangular regions on the left and the right of the origin (= light cones).

Note: in the above diagrams, we drew the time axis from left to right, a direction considered natural by many people; however, physicists prefer to draw the time axis from bottom to top. For that reason, physics books show *vertical* light cones (looking like hour glasses), whereas we showed them horizontal.