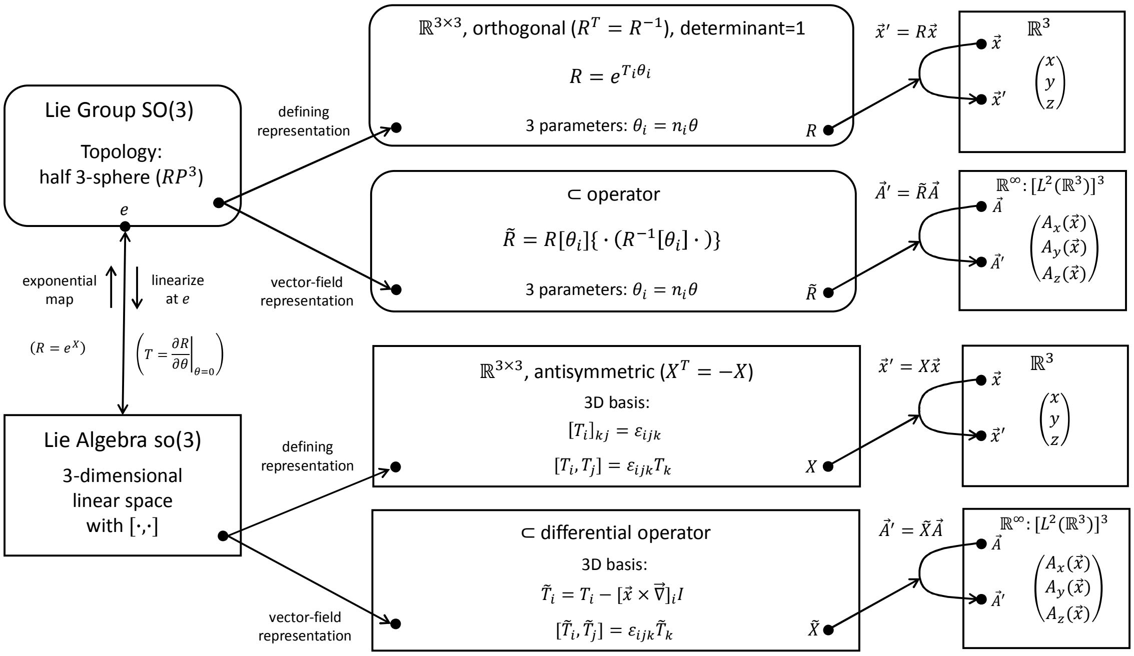
3.29 SO(3): Vector-Field Representation; Angular Momentum of Light



In the following, we have a look at an application of SO(3) to classical field theory. The elements of our representation space are now configurations of the electromagnetic vector potential field in physical 3D space. Since we are now dealing with *vector fields* (as opposed to scalar fields), the SO(3) transformations act not only on the 3D-space coordinates of the field but also on the field components. We know from our spinor-field example that this leads to generators that have two parts. Then, we apply Noether's theorem to these symmetry generators of rotation to find the angular momentum of the electromagnetic field. As expected, we get a two-part result: the orbital angular momentum and the spin. Let's see how this works out in detail!

The electromagnetic vector potential has four components, but thanks to gauge freedom we can set one of them to zero without loss of generality. Here, we choose the temporal gauge, which sets the electric potential to zero, $A_0 = \phi = 0$, leaving only the x-, y-, and z-components of the magnetic vector potential: $\vec{A}(\vec{x}) = (A_x(\vec{x}), A_y(\vec{x}), A_z(\vec{x}))^T$, where $\vec{x} = (x, y, z)^T$ [TM, Vol. 3, Ch. 11.4.5].

Like in the spinor-field example, we'll use a more compact notation to keep our expressions from getting too messy. First, we choose axis-angle rotation parameters $\theta_i = n_i\theta$, where n_i with $i = x, y, z$ is the unit vector along the axis and θ is the rotation angle about this axis. Then, invoking the Einstein summation convention, which implies a summation over repeated indices, we write a general element of the Lie algebra as $X = T_i\theta_i$, where the T_i are the basis generators. The upper branch of the diagram shows again the defining representation of SO(3), but now using our more compact notation.

Following the same steps as in the spinor-field example, but now using the 3-dimensional representation $R[\theta_i]$ to transform real 3D vectors instead of the 2-dimensional representation $D[\theta_i]$ to transform spinors, we find the overall transformation $\vec{A}'(\vec{x}) = R[\theta_i]\,\vec{A}(R^{-1}[\theta_i]\,\vec{x})$ (no summation over i). After splitting off the vector field, the transformation operator by itself is $\tilde{R} = R[\theta_i]\{\cdot(R^{-1}[\theta_i]\cdot)\}$, where, as usual, the dots are place holders for the field and its argument (see the diagram).

To obtain the generators, we take the derivative of $\tilde{R}$ with respect to the θ_i (using the product rule) and then set $\theta_i = 0$. Again, the steps are completely analogous to those of our spinor-field example, except that the generators of D get replaced by generators of R and we don't multiply them by i. We find the three basis generators as $\tilde{T}_i = T_i - [\vec{x} \times \vec{\nabla}]_i I$, where $\vec{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$ and I is the 3×3 identity matrix. For example, the generator $\tilde{T}_z$ for rotation about the z axis evaluates to

$$\tilde{T}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} & -1 & 0 \\ 1 & y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} & 0 \\ 0 & 0 & y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{pmatrix}.$$

This generator has two parts: the first part rotates the vector components and the second part rotates the space coordinates of the field. The generator for an arbitrary axis of rotation is $\tilde{X} = \tilde{T}_i \theta_i$.

Having found the generators of rotation for the electromagnetic potential field, we can now use Noether's theorem to determine the angular momentum of this field. To do that, we need to generalize the conserved quantity $Q = \vec{p} \cdot X\vec{q}$, introduced earlier, from particles to fields. Given a vector field $\vec{\phi}(\vec{x})$ and a linear symmetry transformation acting on it, the conserved quantity is $Q = \int \vec{\pi}(\vec{x}) \cdot \tilde{X}\vec{\phi}(\vec{x})\, d^3x$, where $\tilde{X}$ is the generator of the symmetry and the vector field $\vec{\pi}(\vec{x})$ is the canonical momentum density conjugate to the field $\vec{\phi}(\vec{x})$ [TM, Vol. 3, Ch. 11.4.5]. In contrast to Noether's theorem for particles, the degrees of freedom are now given by the field components ϕ_i at every point in space rather than the position coordinates q_i. Whereas we had only three degrees of freedom for our single-particle example, we now have infinitely many. Note that the space coordinates $\vec{x}$ of the field are now mere labels, just like the indices of the field components. Besides summing over the field components (implicit in the dot product), we also integrate over all space.

Before we can apply Noether's theorem to the electromagnetic potential $\vec{A}(\vec{x})$, we need to determine its canonical momentum density. Given our choice of gauge and working in units where the speed of light is $c = 1$, this turns out to be minus the electric field, $\vec{\pi}(\vec{x}) = -\vec{E}(\vec{x})$ [TM, Vol. 3, Ch. 11.4.5]. Assuming full rotational symmetry (isotropy), there are three conserved quantities, which we call J_i because they are components of angular momentum: $J_i = \int -\vec{E} \cdot (T_i - [\vec{x} \times \vec{\nabla}]_i I)\, \vec{A}\, d^3x = -\int \vec{E} \cdot T_i\vec{A}\, d^3x + \int \vec{E} \cdot [\vec{x} \times \vec{\nabla}]_i\, \vec{A}\, d^3x$. The first term is the spin of the electromagnetic field

$$S_i = -\int \vec{E} \cdot T_i\vec{A}\, d^3x = \int [\vec{E} \times \vec{A}]_i\, d^3x,$$

which is a measure of its *circular polarization* ($\vec{E} \cdot T_i\vec{A} = \vec{E} \cdot \vec{t}_i \times \vec{A} = -\vec{t}_i \cdot \vec{E} \times \vec{A}$, where $\vec{t}_i$ is the i-th standard basis vector). The second term is the orbital angular momentum of the electromagnetic field

$$L_i = \int \vec{E} \cdot \left[\vec{x} \times \vec{\nabla}\right]_i \vec{A}\, d^3x,$$

which is a measure of its *wavefront shape*. For more information on this topic, see the Wikipedia entry "Angular momentum of light".