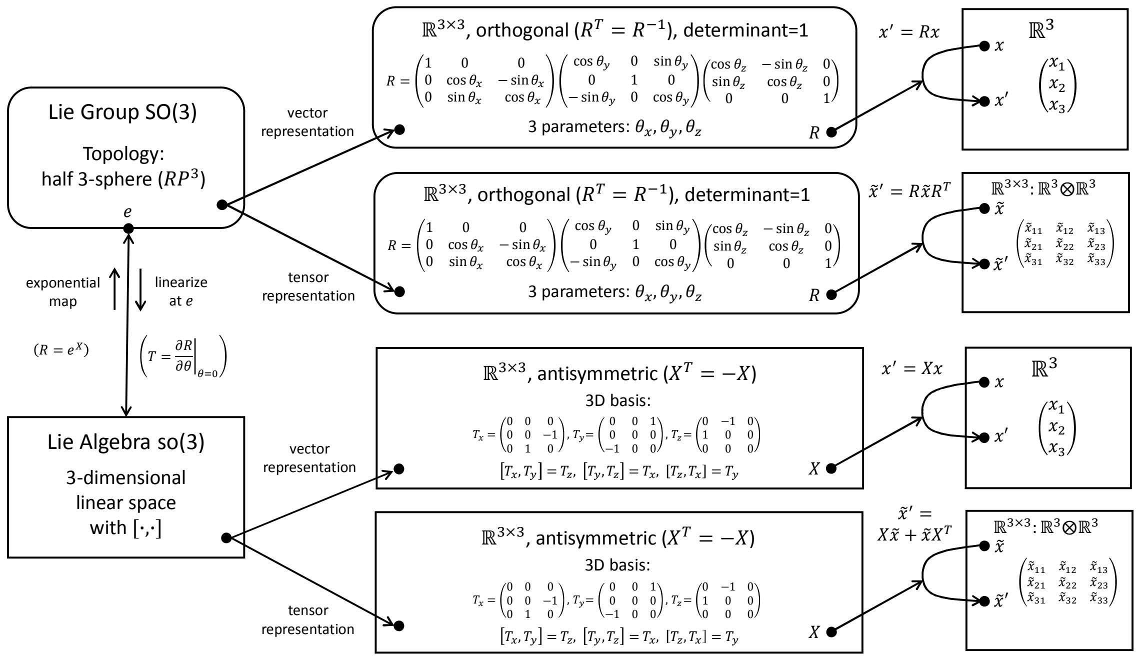
3.30 SO(3): Tensor-Product Representation; Decomposition



Given a representation on vectors, we can construct a representation on tensors. In this example, we take two copies of the defining representation of SO(3) and construct the 9-dimensional tensor-product representation. Then, we'll decompose it into irreducible representations.

The upper branch of the diagram shows again the defining representation of SO(3). For our tensor-product representation we have a choice of writing the elements of the representation space as 3×3 matrices (tensors) or as 9-component column vectors. In the first case, shown in the lower branch of the diagram, the group elements R act on the matrix like $\tilde{x}' = R\tilde{x}R^T$ and the algebra elements X act on the matrix like $\tilde{x}' = X\tilde{x} + \tilde{x}X^T$. In the second case (not shown), the group and algebra elements are 9×9 matrices and act by simple matrix-vector multiplication. This is completely analogous to what we have said about the tensor-product representations of SU(2).

Is this 9-dimensional tensor-product representation reducible? Yes, it is. We know that tensors can be decomposed into a symmetric and an antisymmetric part, which do not "mix" when transformed. Here, the symmetric representation is six dimensional and the antisymmetric representation is three dimensional. Are these two representations irreducible? Based on what we learned from SU(2), we might think so. But this can't be right: we know that SO(3) has only odd-dimensional irreducible representations! It turns out that the symmetric 6-dimensional representation can be broken up once more into a 5- and 1-dimensional irreducible representation. Thus, the compete decomposition of the tensor-product representation is $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$.

Why can we break up the symmetric representation of SO(3), wheras we couldn't do that for SU(2)? Let's look at the trace of the transformed tensor, $\mathrm{tr}(\tilde{x}') = \mathrm{tr}(R\tilde{x}R^T) = \mathrm{tr}(R^T R\tilde{x})$, where in the last step we used the fact that the trace doesn't change under cyclic permutation of its arguments (cyclic property). But what is $R^T R$? For SO(3), it is, by definition, the identity matrix. So, we have $\mathrm{tr}(\tilde{x}') = \mathrm{tr}(\tilde{x})$, that is, the trace remains invariant! In contrast, for SU(2) the expression $U^T U$ is *not* the identity

($U^\dagger U$ would be, but that doesn't help us here) and therefore the trace is *not* invariant. This is why, for SU(2), we had $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$ and not $\mathbf{2}\oplus\mathbf{1}\oplus\mathbf{1}$.

Like for SU(2), we can use the tensor product to find new irreducible representations of SO(3). Specifically, we can take the tensor product of k copies of the 3-dimensional (defining) representation to get a representation on rank-k tensors. Then, we break this 3^k-dimensional representation into irreducibles. For example, for $k = 2$: $\mathbf{3}\otimes\mathbf{3} = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$, as we just showed, and for $k = 3$: $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{3}\otimes(\mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}) = \mathbf{7}\oplus\mathbf{5}\oplus\mathbf{5}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{1}$. As we step through $k = 2, 3, 4, \ldots$, we get a new $(2k + 1)$-dimensional irreducible representation at every step [GTNut, IV.1]! For example, we get $\mathbf{5}$ from $\mathbf{3}\otimes\mathbf{3}$ and we get $\mathbf{7}$ from $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$. It turns out that this new irreducible representation is furnished by the traceless totally symmetric rank-k tensor.

To make the decomposition $\mathbf{3}\otimes\mathbf{3} = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$ more concrete, let's write down possible bases for the three sub-representation spaces. For the 1-dimensional trace sub-representation and the 3-dimensional antisymmetric sub-representation, we may use

$$\tilde{x}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{and} \quad \tilde{x}_{3,a} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \tilde{x}_{3,b} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}, \tilde{x}_{3,c} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and for the 5-dimensional traceless symmetric sub-representation, we may use

$$\tilde{x}_a = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \tilde{x}_b = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \tilde{x}_c = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tilde{x}_d = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \tilde{x}_e = \frac{1}{\sqrt{6}}\begin{pmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Together, these nine matrices form a basis for the full 9-dimensional tensor-product space. An SO(3) transformation acting on this space maps each subspace to itself. (The above basis matrices, $\tilde{x}_i$, are orthogonal and normalized such that $\mathrm{tr}(\tilde{x}_i \tilde{x}_j) = \delta_{ij}$.)

Let's determine the generators of the 5-dimensional sub-representation. A general tensor in this 5-dimensional subspace can be written as $\tilde{x} = a\tilde{x}_a + b\tilde{x}_b + c\tilde{x}_c + d\tilde{x}_d + e\tilde{x}_e$. Now, let's act on this tensor with the generator T_x and determine what happens to the five coefficients a, b, c, d, e. The generator acts like $\tilde{x}' = T_x\tilde{x} + \tilde{x}T_x^T$, that is, $\tilde{x}' = a(T_x\tilde{x}_a + \tilde{x}_a T_x^T) + b(T_x\tilde{x}_b + \tilde{x}_b T_x^T) + c(T_x\tilde{x}_c + \tilde{x}_c T_x^T) + d(T_x\tilde{x}_d + \tilde{x}_d T_x^T) + e(T_x\tilde{x}_e + \tilde{x}_e T_x^T)$. Evaluating the expressions in the parentheses yields $\tilde{x}' = a\left(\sqrt{3}\tilde{x}_e - \tilde{x}_d\right) + b(-\tilde{x}_c) + c(\tilde{x}_b) + d(\tilde{x}_a) + e\left(-\sqrt{3}\tilde{x}_a\right)$. Note that each basis tensor gets mapped to a linear combination of other basis tensors in the *same* 5D subspace (e.g., $\tilde{x}_a$ to $\sqrt{3}\tilde{x}_e - \tilde{x}_d$). Finally, we compare this result with the generic form $\tilde{x}' = a'\tilde{x}_a + b'\tilde{x}_b + c'\tilde{x}_c + d'\tilde{x}_d + e'\tilde{x}_e$ and find that the five coefficients transform as follows: $a' = d - \sqrt{3}e$, $b' = c$, $c' = -b$, $d' = -a$, $e' = \sqrt{3}a$, that is,

$$\begin{pmatrix}d'\\ c'\\ e'\\ a'\\ b'\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & \sqrt{3} & 0\\ 1 & 0 & -\sqrt{3} & 0 & 0\\ 0 & 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}d\\ c\\ e\\ a\\ b\end{pmatrix},$$

where we have arranged the coefficients such that the 5×5 matrix exactly reproduces the 5-dimensional generator $\tilde{T}_x$ from our first SO(3) example! Generators $\tilde{T}_y$ and $\tilde{T}_z$ can be obtained in the same way.