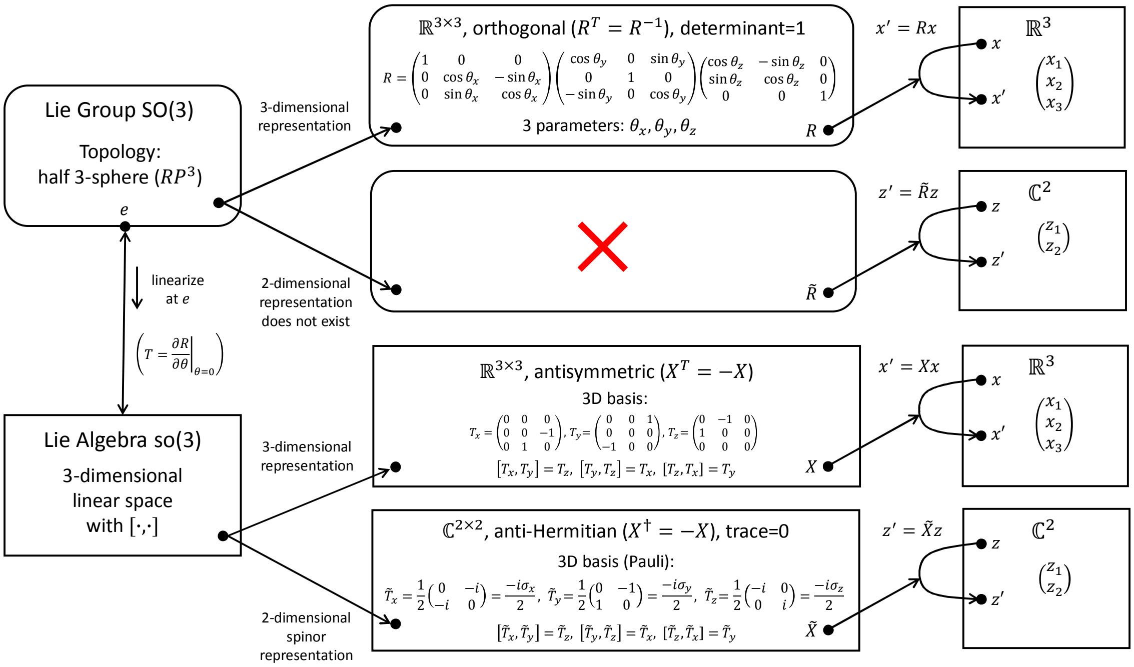
3.22 SO(3): Spinor Representations of the Algebra; Covering Group Spin(3)



Earlier, we said that SO(3) has only odd-dimensional irreducible representations, in particular, it does *not* have a 2-dimensional representation, as indicated with the red × in the diagram. So, if the group doesn't have a 2-dimensional representation, then we can't linearize it at the identity element and therefore the algebra doesn't have a 2-dimensional representation either, right? No, this is not correct!

We can look at a Lie algebra in two different ways. On the one hand, we can view it as the linearization of the corresponding Lie group (its tangent space) together with a Lie-bracket operation that measures the degree to which the group elements fail to commute. On the other hand, we can view it as an abstract algebra that needs to satisfy certain axioms plus the commutation relations obtained from the first view. More specifically, the axioms define a vector space with a Lie-bracket operation satisfying bi-linearity, anti-commutativity, and the Jacobi identity [PfS, Ch. 3.4.2]. If we look at the Lie algebra from the second point of view, then the so(3) algebra does have a 2-dimensional representation! We already know what it is. Since so(3) has the same commutation relations as su(2), the 2-dimensional representation of so(3) is just the same as that of su(2) (see the lower half of the diagram). The same is true for all the other even-dimensional representations of so(3). These representations are known as the *spinor representations* of the so(3) algebra.

Since SO(3) describes rotations in our familiar 3D space, we expect its representations to act on real vectors and keep them real. This is indeed what the odd-dimensional representations of the group as well as the algebra do. However, the even-dimensional representations of the algebra force us to consider complex vector spaces. This may seem strange at first, but there is nothing wrong about a complex representation space. If we want to find all the irreducible representations of so(3), then we have to accept complex vectors. This is analogous to if we want to find all n roots of a general polynomial of degree n, then we have to accept complex roots.

We discovered the spinor representations of so(3) easily because we already knew about the representations of su(2). In the general case of so(n), we can find the spinor representations by starting with the *Clifford algebra* over an n-dimensional vector space with Euclidean metric, Cliff(n,0). The basis generators of the spinor representation of so(n) then appear as second-order elements ($\frac{1}{2}\gamma_i\gamma_k$) of this algebra [GNut, Ch. VII.1]. We will see an example of this when we come to the Dirac spinors. Whereas the lowest spinor representation of so(3) is 2-dimensional, we will see that that of so(4) (or so(1,3)) is 4-dimensional and splits into two 2-dimensional irreducible representations, which are related by space inversion. Similarly, we will see that so(2) has two 1-dimensional spinor representations related in the same way. All spinor representations act on complex vectors. The general pattern is:

Orthogonal Algebra	Spinor Dimension(s)
so(2)	1+1
so(3)	2
so(4)	2+2
so(5)	4
so(6)	4+4
so(7)	8
so(8)	8+8
so(9)	16
so(10)	16+16

What happens if we take the exponential map of a spinor representation of so(3)? Well, we don't get a representation of SO(3) because SO(3) doesn't have even-dimensional representations. What we do get is a representation of a slightly different Lie group, namely the *double cover* of SO(3). We already know what this group is: it is our old friend SU(2)! More generally, exponentiating the algebra so(n) takes us to the *covering group* known as Spin(n). It just so happens that Spin(3) is isomorphic to SU(2).

The covering group plays an important role in quantum mechanics. Quantum states are described by vectors in Hilbert space *up to a phase*. In other words, they are equivalence classes in Hilbert space. For this reason, we are interested in group representations on Hilbert space *up to a phase*. Such representations are known as *projective representations*. Now, it turns out that the projective representations of a group are equivalent to the regular representations of the covering group [GFKG, Ch. II.1, p. 182]! Thus, if the classical symmetry is SO(3), the relevant quantum symmetry is SU(2). This brings us full circle to where we started: the rotational symmetry of quantum spin.

We have seen in this example that the representations of the group SO(3) and the algebra so(3) are not in a one-to-one relationship like they were for SU(2) and su(2). Why is that? It turns out that this has a topological reason: the manifold of SO(3) is *not simply connected*, whereas the manifold of SU(2) (= 3-sphere) was simply connected. Simply connected means that we can pick any loop in the group manifold and contract it to a point in a continuous manner without getting stuck. A Lie algebra (e.g., so(3)) may be associated with more than one Lie group (e.g., SO(3) and Spin(3)), but only *one* of them is *simply connected* (Spin(3), in this case). This simply-connected group is the *covering group* and it is the one obtained by exponentiating the algebra [PfS, Ch. 3.4.4]. In the next example, we'll discuss the topology of SO(3).