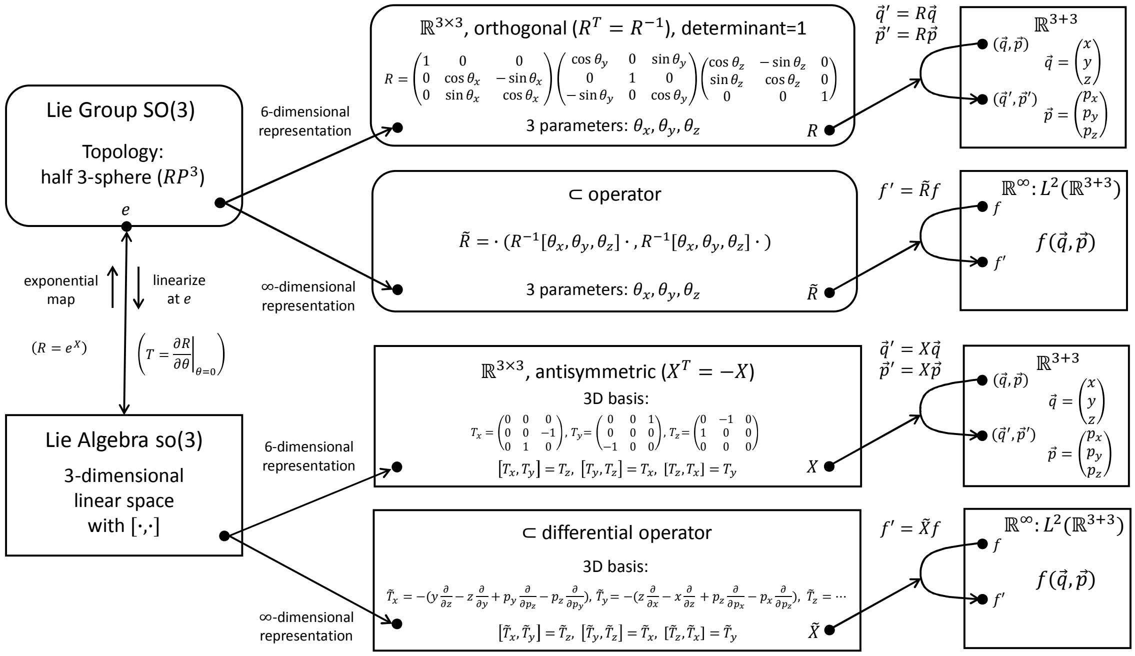
3.27 SO(3): Infinite-Dimensional Representation on Phase-Space Functions



Next, we take the *phase space* known from classical Hamiltonian mechanics as our representation space. A point (q,p) in this space is given by the (generalized) position coordinates *and* the canonical momentum coordinates. It describes the state of the classical system. A transformation that acts on such a state and leaves the Hamiltonian formalism intact is known as a *canonical transformation* [NNCM, Ch. 7.3]. Here, we focus on a single particle in 3D space and thus we have $\vec{q} = (x,y,z)^T$ and $\vec{p} = (p_x,p_y,p_z)^T$ together defining a 6-dimensional phase space. Furthermore, we take as the canonical transformation $\vec{q}' = R\vec{q}$ and $\vec{p}' = R\vec{p}$, which rotates the state of the particle about a given axis. (The fact that $\vec{p}$ and $\vec{q}$ are transformed by the same matrix is a consequence of R being orthogonal.) See the upper branch of the diagram.

Now, let's consider (scalar) functions on our 6-dimensional phase space: $f(\vec{q},\vec{p})$. As we know from the previous example, such functions represent classical observables such as the angular momentum of the particle about the z axis $f(\vec{q},\vec{p}) = xp_y - yp_x$. Additional examples of *phase-space functions* are the distance of the particle from the origin $f(\vec{q},\vec{p}) = \sqrt{x^2+y^2+z^2}$, the momentum of the particle in the x direction $f(\vec{q},\vec{p}) = p_x$, and the kinetic energy of the particle $f(\vec{q},\vec{p}) = \frac{1}{2}(p_x^2+p_y^2+p_z^2)/m$. Not surprisingly, functions like these furnish an infinite dimensional representation of SO(3).

How does SO(3) act on these phase-space functions? It acts by multiplying the two 3D arguments $\vec{q}$ and $\vec{p}$ by the inverse of the 3×3 rotation matrix: $f'(\vec{q},\vec{p}) = f(R^{-1}[\theta_x,\theta_y,\theta_z]\vec{q}, R^{-1}[\theta_x,\theta_y,\theta_z]\vec{p})$. We discussed this kind of transformation earlier for the infinite-dimensional representation of SU(2). How can we write this transformation as a disembodied operator $\tilde{R}(\theta_x,\theta_y,\theta_z)$ that acts on the function $f(\vec{q},\vec{p})$? Using our informal dot notation, we write $\tilde{R}(\theta_x,\theta_y,\theta_z) = \{\cdot\,(R^{-1}[\theta_x,\theta_y,\theta_z]\,\cdot\,, R^{-1}[\theta_x,\theta_y,\theta_z]\,\cdot\,)\}$, where the first dot is a place holder for the function's name, the second dot is a place holder for the first argument, and the third dot is a place holder for the second argument. See the lower branch of the diagram.

What are the elements $\tilde{X}$ of our new Lie-algebra representation? Differentiating $\tilde{R}(\theta_x, \theta_y, \theta_z)\, f(\vec{q}, \vec{p}) = f(R^{-1}\vec{q}, R^{-1}\vec{p})$ with respect of its parameters and evaluating the result at the identity element yields $\tilde{X} f(\vec{q}, \vec{p}) = \vec{\nabla}_q f(\vec{q}, \vec{p}) \cdot [-X\vec{q}] + \vec{\nabla}_p f(\vec{q}, \vec{p}) \cdot [-X\vec{p}]$, where we made use of the chain rule and wrote the generator of R as X, which we already know from the representation in the upper branch. Now, splitting off the operator from the function, we get the differential operator

$$\tilde{X} = -\big(X\vec{q} \cdot \vec{\nabla}_q + X\vec{p} \cdot \vec{\nabla}_p\big).$$

For example, for $X = T_z$, we find

$$\tilde{T}_z = -\left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial p_x \\ \partial/\partial p_y \\ \partial/\partial p_z \end{pmatrix}\right]$$
$$= -\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x} + p_x\,\frac{\partial}{\partial p_y} - p_y\,\frac{\partial}{\partial p_x}\right).$$

The diagram shows the remaining two basis generators. If we pack all three basis generators into a vector, we can write more compactly $(\tilde{T}_x, \tilde{T}_y, \tilde{T}_z)^T = -(\vec{q} \times \vec{\nabla}_q + \vec{p} \times \vec{\nabla}_p)$. These basis generators are essentially the same ones we found earlier for the infinite-dimensional representation of SU(2), except that now they also act on the momentum coordinates.

Note that the equation $\tilde{X} = -(X\vec{q} \cdot \vec{\nabla}_q + X\vec{p} \cdot \vec{\nabla}_p)$ tells us how to “upgrade” the generator X, acting on phase-space coordinates, to the generator $\tilde{X}$, acting on phase-space functions. This will become important when we come to the Poisson brackets in the next example.

To check whether an operator is orthogonal, we test if $\int \tilde{R} f(x) \tilde{R} g(x)\, dx = \int f(x) g(x)\, dx$ holds for any pair of functions $f(x)$ and $g(x)$. (This is analogous to the matrix test $(Rx)^T Ry = x^T y$ from which $R^T R = I$ follows.) Similarly, to check whether an operator is antisymmetric, we test if $\int f(x) \tilde{X} g(x)\, dx = -\int \tilde{X} f(x) g(x)\, dx$ holds for any pair of functions $f(x)$ and $g(x)$. (This is analogous to the matrix test $x^T X y = -(Xx)^T y$ from which $X = -X^T$ follows.) It turns out that our rotation operator $\tilde{R}$ is orthogonal and our differential operator $\tilde{X}$ is antisymmetric.

In quantum mechanics, the eigenvectors (or eigenfunctions) of the generators played an important role: they represented the states for which the corresponding observable has a definite value. In classical mechanics all states have a definite value and the eigenvectors (or eigenfunctions) of the generators merely characterize the vector field that is associated with the transformation. For example, the basis generator T_z has eigenvectors $(i, 1, 0)^T$, $(-i, 1, 0)^T$, and $(0, 0, 1)^T$ with eigenvalues i, $-i$, and 0, respectively. From the last eigenvector/eigenvalue pair we conclude that the vector field has no flow in the z direction.