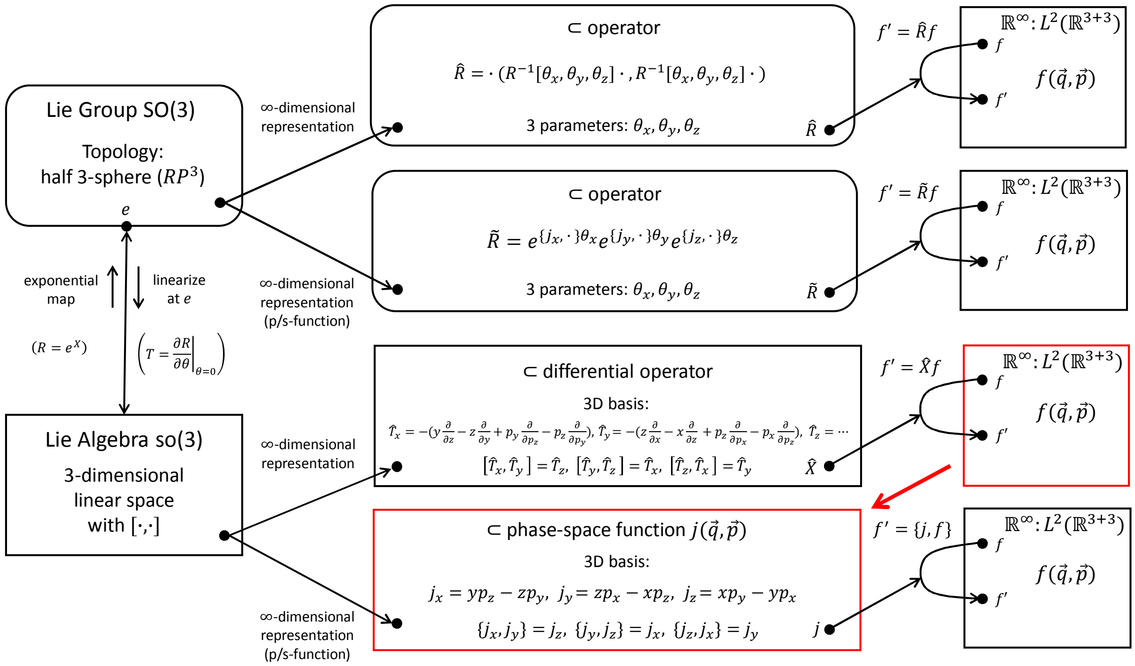
3.28 SO(3): Representation with Functions as Generators; Poisson Bracket



We saw earlier that Noether's theorem associates symmetry generators with conserved phase-space functions. For example, the generator T_z of so(3) is associated with the (single-particle) phase-space function $j_z(\vec{q},\vec{p}) = xp_y - yp_x$. This observation allows us to reformulate the representation from the previous example (shown again in the upper branch of the diagram) such that the Lie-algebra elements become phase-space functions (lower branch of the diagram). Now, the generators do not only act *on* phase-space functions but they *are* phase-space functions themselves (red arrow in the diagram). But how do these "generator functions" act on the functions in the representation space?

From the last two examples we know the following: First, given a generator X acting on phase space, the associated (single-particle) phase-space function is $g(\vec{q},\vec{p}) = \vec{p}\cdot X\vec{q}$. Second, given a generator X acting on phase space, we can "upgrade" it to the generator $\hat{X} = -(X\vec{q}\cdot\vec{\nabla}_q - X\vec{p}\cdot\vec{\nabla}_p)$ acting on (single-particle) phase-space *functions*. Now, we combine these two facts and express $\hat{X}$ in terms of g. Taking the gradient of the phase-space function $g(\vec{q},\vec{p}) = \vec{p}\cdot X\vec{q}$ with respect to $\vec{p}$, we get $\vec{\nabla}_p g = X\vec{q}$, which is a piece of the expression for $\hat{X}$. Rewriting $g(\vec{q},\vec{p}) = \vec{q}\cdot X^T\vec{p}$, and taking the gradient with respect to $\vec{q}$, we get $\vec{\nabla}_q g = X^T\vec{p}$. Using of the antisymmetry of X, we can write $\vec{\nabla}_q g = -X\vec{p}$, which is the second piece of the expression for $\hat{X}$. Inserting the two pieces into the expression for $\hat{X}$ yields $\hat{X} = -\vec{\nabla}_p g\cdot\vec{\nabla}_q + \vec{\nabla}_q g\cdot\vec{\nabla}_p$, which expresses the generator $\hat{X}$ in terms of the phase-space function g, as desired. When acting with this generator on another phase-space function f, we get

$$\hat{X}f = \vec{\nabla}_q g\cdot\vec{\nabla}_p f - \vec{\nabla}_p g\cdot\vec{\nabla}_q f = \sum_i\left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right),$$

which is exactly the definition of the *Poisson bracket*: $\{g,f\}$! We have thus succeeded in reformulating the infinite-dimensional representation of so(3) such that the Lie-algebra elements $\hat{X}$ become phase-space functions $g(\vec{q},\vec{p})$. The generator functions g act on the representation-space functions f by means of the Poisson bracket $f' = \{g,f\}$. Equivalently, we can write $f' = \hat{X}f$, where $\hat{X} = \{g,\ \cdot\}$.

Let's try this out for the generator function $j_z(\vec{q},\vec{p}) = xp_y - yp_x$. It acts on phase-space functions like $\{xp_y - yp_x, \cdot\}$, which evaluates to

$$\frac{\partial(xp_y - yp_x)}{\partial x}\frac{\partial}{\partial p_x} + \frac{\partial(xp_y - yp_x)}{\partial y}\frac{\partial}{\partial p_y} - \frac{\partial(xp_y - yp_x)}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial(xp_y - yp_x)}{\partial p_y}\frac{\partial}{\partial y}$$
$$= p_y\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_y} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

This matches exactly the differential operator $\hat{T}_z$, as expected. For the other two basis-generator functions, j_x and j_y, we get analogous results.

Next, we need to find the appropriate Lie-bracket operation for our generator functions. In the case of the differential operators, $\hat{T}_n$, it was the commutator. What is the corresponding operation for generator functions, $j_n(\vec{q},\vec{p})$? As we might guess from our experience with the cross product, it is the Poisson bracket again! To show this, we evaluate $[\hat{T}_m, \hat{T}_n] = \hat{T}_m\hat{T}_n - \hat{T}_n\hat{T}_m = \{j_m, \{j_n, \cdot\}\} - \{j_n, \{j_m, \cdot\}\} = \{\{j_m, j_n\}, \cdot\}$, where we used the Jacobi identity in the last step.

Can we write the elements of the Lie *group* in terms of the generator functions introduced above? Yes, we exponentiate the generators, as usual! For example, the basis generator $j_z(\vec{q},\vec{p})$ produces the transformation $\tilde{R} = \exp(\{j_z(\vec{q},\vec{p}), \cdot\}\theta_z)$, which describes a rotation of a phase-space function by the angle θ_z about the z axis (see the lower branch of the diagram). To make sense of this exponential, we let it act on the phase-space function $f(\vec{q},\vec{p})$ and expand it into the power series $f' = \tilde{R}f = f + \{j_z, f\}\theta_z + \frac{1}{2}\{j_z\{j_z, f\}\}\theta_z^2 + \cdots$.

The Poisson bracket is well known from classical Hamiltonian mechanics [TM, Vol. 1, Ch. 10; NNCM, Ch. 5]. In the Hamiltonian framework, phase-space functions play *two* roles. First, they represent the *observables* of the theory and second, they represent *generators* of transformations. In the latter case, they act on observables by means of the Poisson bracket. In our example, the three phase-space functions j_x, j_y, and j_z are the observables for the x-, y-, and z-component of the angular momentum. When put into Poisson brackets, they generate rotations about the x-, y-, and z-axis.

In the Hamiltonian framework, the time evolution of an observable f is given by the differential equation $\partial f/\partial t = \{f, h\}$, where h is the Hamiltonian function, which acts as the generator of time translation. Thus, an observable is conserved (= time independent) if it "Poisson commutes" with the Hamiltonian: $\{f, h\} = 0$. Furthermore, a generator f acts on the system like $\{h, f\}$, where h is again the Hamiltonian function, but now representing the system's law of time evolution (which also happens to be the energy observable). Thus, a generator generates a symmetry (= law of time evolution is invariant) if it "Poisson commutes" with the Hamiltonian: $\{h, f\} = 0$. In other words, a generator of a symmetry and its associated conserved observable are represented by the *same* phase-space function $f(q, p)$. This feature of Hamiltonian mechanics motivated the representation discussed in this example. See the Appendix "Symmetry and Conservation in Classical Hamiltonian Mechanics" for more information.