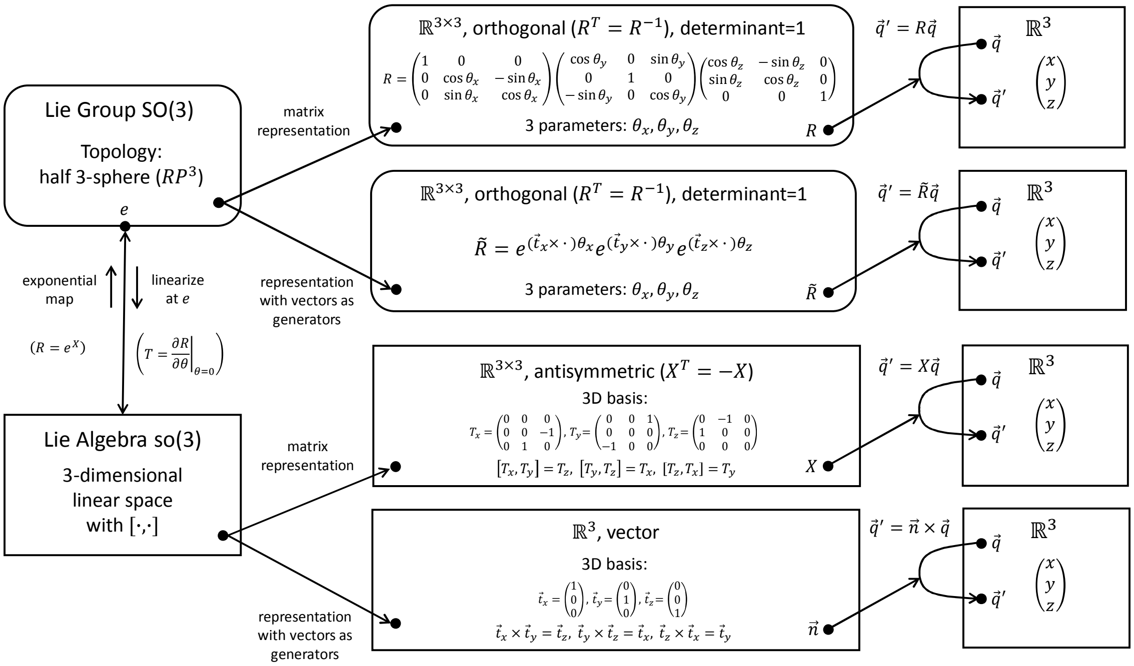
3.26 SO(3): Application to Classical Angular Momentum; Noether's Theorem



Next, let's have a look at an application of SO(3) to classical mechanics. In classical mechanics, the (generalized) position coordinates, which represent the degrees of freedom of the system, define a point, q, in *configuration space*. A transformation that acts on such a point in configuration space is known as a *point transformation* [NNCM, Ch. 7.2]. Here, we focus on a single particle in 3D space and thus q is the 3D vector $\vec{q} = (x, y, z)^T$. Furthermore, we take as the point transformation $\vec{q}\,' = R\vec{q}$, which rotates the position of the particle about a given axis. We know from the previous example that the generator of such a transformation can be written as either a matrix X (upper branch of the diagram) or an axis vector $\vec{n}$ (lower branch of the diagram).

Noether's theorem says that for every continuous symmetry there is a conserved quantity. For the case where the symmetry is given by a linear point transformation acting on a single particle, the conserved quantity (= Noether charge) is given by $Q = \vec{p} \cdot X\vec{q}$, where X is the generator of the symmetry transformation, $\vec{q} = (x, y, z)^T$ is the position of the particle, and $\vec{p} = (p_x, p_y, p_z)^T$ is the canonical momentum (conjugate to $\vec{q}$) of the particle. This formula arises from classical Lagrangian mechanics, which derives the equations of motion from an action principle. For more details, see the Appendices "Symmetry and Conservation in Classical Lagrangian Mechanics" and "Noether's Theorem for Particle Theories" or [TM, Vol. 1, Ch. 7; NNCM, Ch. 10.4]. In Newtonian mechanics, which starts from the equations of motion (without the use of an action principle), this relationship is not at all obvious.

What is the conserved quantity Q for the case of 3D rotational symmetry? We can write an arbitrary generator of 3D rotation as $X = n_x T_x + n_y T_y + n_z T_z$, where the T_i are the basis generators of SO(3) and $\vec{n} = (n_x, n_y, n_z)^T$ is the unit vector pointing along the axis of rotation. Thus, we have $Q = \vec{p} \cdot X\vec{q} = \vec{p} \cdot (n_x T_x + n_y T_y + n_z T_z)\vec{q}$. From the previous example, we know that this can be rewritten more compactly using the cross product as $Q = \vec{p} \cdot (\vec{n} \times \vec{q})$. The entire expression is the *scalar triple product* of the vectors $\vec{p}$, $\vec{n}$, and $\vec{q}$, which can be rewritten again as $Q = \vec{n} \cdot (\vec{q} \times \vec{p})$. Finally, this expression can be interpreted as the projection of the *classical angular momentum*, $\vec{q} \times \vec{p}$, on the axis of rotational symmetry, $\vec{n}$, or more succinctly, the angular momentum (component) in the direction of the axis of rotational symmetry: $j_{\vec{n}} = \vec{n} \cdot (\vec{q} \times \vec{p})$! For example, if the axis of rotational symmetry points in the z direction, $\vec{n} = (0, 0, 1)^T$, the conserved quantity is the angular momentum along the z axis: $j_z = xp_y - yp_x$. If the symmetry holds for any axis $\vec{n}$ (for an isotropic system), the entire angular-momentum vector $\vec{j} = \vec{q} \times \vec{p}$ is conserved, that is, the three quantities j_x, j_y, and j_z are conserved individually.

Note that the conserved quantity $Q = \vec{p} \cdot X\vec{q}$ is a function of position $\vec{q}$ *and* momentum $\vec{p}$. Together, $\vec{q}$ and $\vec{p}$ define the state of the classical system. The space of all possible states, $(\vec{q}, \vec{p})$, is known as the *phase space*. Thus, Noether's theorem tells us how to construct a conserved *phase-space function* from a given symmetry generator: $Q(\vec{q}, \vec{p}) = \vec{p} \cdot X\vec{q}$. For example, the angular momentum $j_z(\vec{q}, \vec{p})$ is the conserved phase-space function that follows from the symmetry generator T_z:

$$j_z(\vec{q}, \vec{p}) = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = xp_y - yp_x.$$

Incidentally, we can reverse this procedure and extract the symmetry generator from a conserved phase-space function by calculating $X = (\vec{\nabla}_q(\vec{\nabla}_p Q(\vec{q}, \vec{p}))^T)^T$, where $\vec{\nabla}_q = (\partial/\partial x, \partial/\partial y, \partial/\partial z)^T$ and $\vec{\nabla}_p = (\partial/\partial p_x, \partial/\partial p_y, \partial/\partial p_z)^T$. For example, given the phase-space function $j_x(\vec{q}, \vec{p}) = yp_z - zp_y$, we find

$$X = (\vec{\nabla}_q(\vec{\nabla}_p j_x(\vec{q}, \vec{p}))^T)^T = (\vec{\nabla}_q(0, -z, y))^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

which is exactly the basis generator T_x.

We are now in a position to compare quantum-mechanical and classical angular momentum. Both derive from the generator of 3D space rotation, but there are some important differences:

- In *quantum mechanics,* the generator is multiplied by $i\hbar$ and the observed value is one of its eigenvalues. These eigenvalues are discrete (quantized) and which one is observed depends probabilistically on the state ψ of the system.
- In *classical mechanics*, the generator is mapped to a phase-space function. The observed value is given by this function evaluated for the state (q, p) of the system. The observed value is continuous and deterministic.

If we are interested in the *expectation value* of a quantum observable only, we can describe it by a (continuous) function of ψ. For example, the expectation value of the z-component of the angular momentum is $\langle J_z \rangle = \psi^\dagger J_z \psi$, which is analogous to the classical expression $j_z = p^T T_z q$.