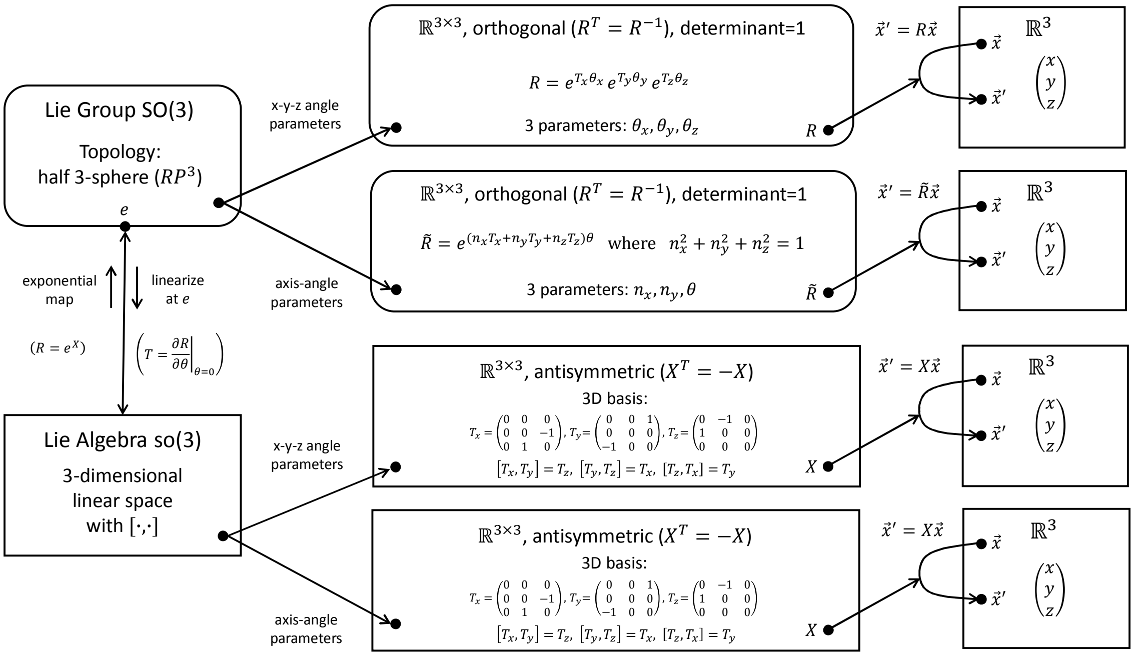
3.23 SO(3): Defining Representation with Axis-Angle Parameters; Group Topology



Like for SU(2), there is more than one way to parametrize an SO(3) matrix. The upper branch of the diagram shows again the defining representation of SO(3) written as the product $R = R_x(\theta_x)\cdot R_y(\theta_y)\cdot R_z(\theta_z) = \exp(T_x\theta_x)\cdot\exp(T_y\theta_y)\cdot\exp(T_z\theta_z)$, corresponding to a rotation about the z axis followed by a rotation about the y axis and finally a rotation about the x axis. Here, the parameters are the rotation angles θ_x, θ_y, and θ_z. As usual, the transformation acts on the column vector $\vec{x} = (x_1, x_2, x_3)^T = (x, y, z)^T$, where the arrow on top of the x indicates that we are dealing with a vector in 3-dimensional Euclidean space (see the upper branch of the diagram).

Now, we are going to specify rotations by the axis of rotation and the rotation angle about this axis. Thus, the new parameters are the unit vector along the axis, $\vec{n} = (n_x, n_y, n_z)^T$, where $n_x^2 + n_y^2 + n_z^2 = 1$, and the rotation angle θ about this axis. There are again three independent parameters, which we list in the lower branch of the diagram as n_x, n_y, θ, implying that n_z is determined by the unit-vector constraint (this is not exactly true because the sign of n_z remains ambiguous). Note that it is a special feature of 3D (and 2D) rotation that any composition of rotations can always be expressed as a single rotation about some axis [RtR, Ch. 11.4].

How do we find the SO(3) matrix with this parametrization? We follow the same procedure as for SU(2). First, we construct the generator of rotation about the axis of interest by linearly combining the basis generators for x, y, and z rotations with the components of the axis vector as coefficients: $X = n_xT_x + n_yT_y + n_zT_z$. Then, we exponentiate this generator to obtain the rotation matrix for an arbitrary angle θ, that is, $\tilde{R} = e^{X\theta} = \exp([n_xT_x + n_yT_y + n_zT_z]\theta)$. In contrast to the SU(2) case, the matrix exponential here does not evaluate to a neat expression.

What is the topology of the SO(3) manifold? The axis-angle parametrization makes clear that every rotation can be associated with a point in a (solid) 3D ball: the line connecting the ball's center with the point in question defines the axis of rotation and the distance between the ball's center and the point defines the angle of rotation (0° for the point at the center to 180° for the point at the surface of the ball). In other words, if we multiply the unit vector $\vec{n}$ for the axis with the rotation angle θ, we get the vector $\vec{n}\theta$, which points to the point in the ball that is associated with the rotation given by $\vec{n}$ and θ. But there is a complication: rotating by 180° clockwise and rotating by 180° counterclockwise (about the same axis) is the *same* transformation! Thus, SO(3) does not have the topology of a regular 3D ball, but that of a 3D ball with antipodal points on its surface identified. Note that, unlike a regular ball, this 3D space has no boundary: coming out of the ball at one point automatically takes us to the other side of the ball without experiencing any discontinuity. You may imagine a "wormhole" going from one point to the other, such that when arriving at one point, we are immediately transported to the other.

Earlier, we discovered that the topology of SU(2) can be understood as two 3D balls that are connected (identified) at their surfaces. Then, we learned that a pair of elements of SU(2), a rotation and the same rotation plus 360°, correspond to a single element of SO(3). One element of these pairs is located in the first and the other one in the second ball. Thus, when mapping from SU(2) to SO(3), the two balls merge into a single ball (with the appropriate surface identifications). It all makes sense!

Here is another way of visualizing the topology of SO(3). We know that SU(2) has the topology of a 3-sphere, S^3. Applying the two-to-one map from SU(2) to SO(3), we find that SO(3) has the topology of a 3-sphere with antipodal points identified, $S^3/\mathbb{Z}_2$. Equivalently, we can describe this topology as just *half* of a 3-sphere with antipodal points on the rim identified. To visualize this space, we descent from four to three dimensions and picture a regular sphere in 3D space that is cut in half at the equator. Then, we identify diametrically opposite points on the equator (which is now the rim). Finally, we “imagine” doing the same construction for a 3-sphere embedded in 4D space.

A very different way of understanding the topology of SO(3) is to associate its group elements not with points but with straight lines (1-dimensional subspaces) in a 4D space. To get familiar with this idea, let's start with a 2D space and consider all the lines that go through the origin. Then, imagine a circle around the origin. Each line through the origin can be identified with a point on the circle, namely the point of intersection. But wait a minute, each line intersects the circle in *two* (diametrically opposite) points!) So, the space of lines really corresponds to only a half-circle and there is a single line corresponding to the two end points of the half circle. This is exactly what we want! Such a space of lines is known as the 1-dimensional *real projective space*, RP^1 (realized as 1D subspaces in a 2D space) [RtR, Ch. 15.6]. Now, we ascend from two to three dimensions and imagine lines in a 3D space passing through the origin. We can identify these lines with the points on the half-sphere discussed above. This is RP^2 (realized as 1D subspaces in a 3D space). Finally, ascending from three to four dimension, we arrive at RP^3, the topology of SO(3)!

To conclude this example, let’s compare SO(3) with SU(2). Whereas SU(2) describes unfamiliar spinorial rotations, it has a clean and simple topology (the 3-sphere, which is simply connected) and has representations in all dimensions from one to infinity. In contrast, whereas SO(3) describes the rotations that make intuitive sense to us, it has an ugly topology (half of a 3-sphere, which is not simply connected) and has only odd-dimensional representations. Since spinorial objects, such as electrons, exist, it seems that Nature chooses mathematical elegance over what makes intuitive sense to us!