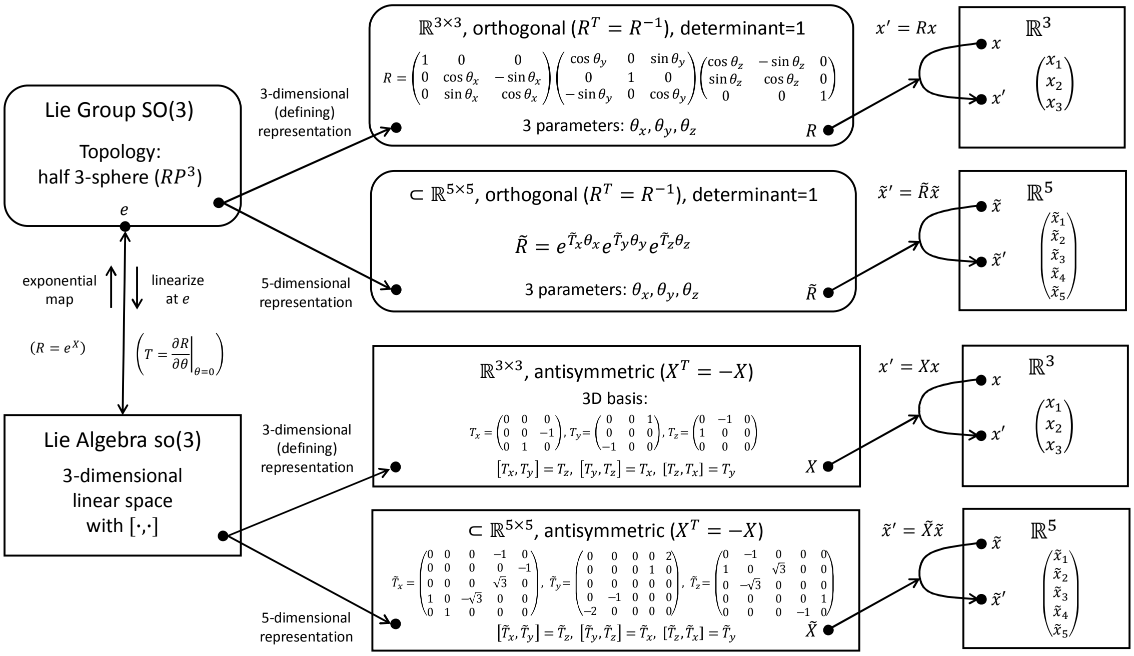
3.21 SO(3): The Group of Rotations in 3-Dimensional Euclidean Space



Let's move on to the Lie group SO(3)! This group, or rather its defining (3-dimensional) representation, consists of all 3×3 orthogonal matrices R with determinant one. The S in SO(3) stands for *special*, indicating that $\det(R) = 1$ and the O stands for *orthogonal*, meaning that the real matrices R satisfy $R^T R = I$. Note that an orthogonal matrix is a unitary matrix ($U^\dagger U = I$) that is also real ($U^\dagger = U^T$). As we will see shortly, SO(3) is closely related to our old friend SU(2).

We know that the unitary transformations preserve the Hermitian inner product. What about the orthogonal transformations? They preserve the *Euclidean inner product* (or dot product) of two vectors: $(Rx)^T Ry = x^T R^T R y = x^T y$. As a consequence, lengths of vectors and angles between vectors remain invariant. Moreover, the origin stays in place because the transformations are linear. Therefore, the orthogonal transformations are simply 3D rotations and/or reflections about the origin. The additional constraint "determinant one" eliminates the reflections, leaving us with only the *proper* rotations. We may parametrize SO(3) transformations by the three rotation angles θ_x, θ_y, and θ_z, as we did for SU(2). SO(3) transformations, which rotate 3D vectors, are much easier to visualize than SU(2) transformations, which were churning two complex numbers!

Note that it is just a coincidence that SO(3) is a *three-dimensional* manifold (= described by three parameters) and its defining representation is also *three dimensional* (= acts on 3D vectors). In general, these two dimensions have nothing to do with each other!

There are several ways of parametrizing an SO(3) matrix. One possibility is to write the matrix $R(\theta_x, \theta_y, \theta_z)$ as a product of three matrices, where each matrix depends on only a single parameter: $R_x(\theta_x) \cdot R_y(\theta_y) \cdot R_z(\theta_z)$, as shown in the diagram (upper branch). In this case, R first rotates about the z axis, then about the y axis, and finally about the x axis, but parametrizations that use other sequences are equally valid. This matrix looks familiar to us: we encountered it before as the 3-dimesional representation of SU(2) acting on real vectors!

To find the basis generators of the corresponding Lie algebra, so(3), we take the derivatives of the transformation matrix with respect to the parameters θ_i and evaluate the results at $\theta_i = 0$. Since we are now focusing on classical mechanics and are no longer interested in the generators being Hermitian, we *don't* multiply the result by i. The resulting basis generators are the real antisymmetric matrices T_x, T_y, and T_z shown in the diagram. These matrices can be expressed in terms of the 3-dimensional Levi-Civita symbol as $[T_i]_{kj} = \varepsilon_{ijk}$, where i labels the matrix (T_1, T_2, T_3 being the same as T_x, T_y, T_z) and k, j label the matrix component. The commutation relations among the three basis generators are $[T_x, T_y] = T_z$, $[T_y, T_z] = T_x$, and $[T_z, T_x] = T_y$, which is exactly the same that we had for su(2) before we multiplied the generators by i! In fact, the so(3) and su(2) Lie algebras are *isomorphic*.

What happens if we rotate an object by a small amount about the x axis, then rotate it by a small amount about the y axis, then undo the rotation about the x axis, and finally undo the rotation about the y axis in that order? No, we are not back to where we started. We end up with a small rotation about the z axis! This is the meaning of the commutation relation $[T_x, T_y] = T_z$.

Although the so(3) and su(2) Lie algebras are isomorphic, the corresponding Lie groups, SO(3) and SU(2), are not! For SO(3) a 360° rotation is the same thing as doing nothing (the identity transformation). But, as we know, for SU(2) we need a 720° rotation to get back to where we started. In fact, for every element in SO(3) there are two corresponding elements in SU(2): a rotation by θ and a rotation by $\theta + 360°$. We say that SU(2) *double covers* SO(3). SO(3) and SU(2) are locally isomorphic but globally different.

Like SU(2), SO(3) has infinitely many representations. Besides the trivial 1-dimensional representation (which always exists) and the defining 3-dimensional representation, there is also a 5-dimensional representation (see the lower branch of the diagram), a 7-dimensional representation, etc. (cf. http://visuallietheory.blogspot.com/2013/). However, unlike SU(2), SO(3) *doesn't* have any even-dimensional irreducible representations! How is the 5-dimensional representation shown in the diagram related to the 5-dimensional (spin-2) representation of SU(2), which we encountered earlier? They are related by a similarity transformation and thus are equivalent.

The S in the group names SO(3) and SU(2) indicates that the transformation matrix has determinant one. Yet, the effects of this condition on SO(3) and SU(2) are quite different. The group U(2), without the S, has four basis generators. But because one generator commutes with all the others, it is natural to break the group up into a 1-dimensional group, U(1), and a 3-dimensional group, SU(2), which then can be dealt with separately. In fact, we may write, somewhat sloppily, U(2) = SU(2) × U(1) (the mathematically correct formula is U(2) = (SU(2)/$\mathbb{Z}_2$) × U(1) [GTNut, p. 253]). We will discuss the group U(1) later.

On the other hand, the group O(3), without the S, consists of two *components*: the proper (= non-reflected) rotations and the improper (= reflected) rotations. The elements in the second component cannot be reached in a continuous manner from the identity element. Only by eliminating the second component, that is, by restricting to SO(3), do we get a *connected* Lie group. We may write O(3) = SO(3) × $\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the cyclic group with two elements: the identity element and the reflection element. We will discuss $\mathbb{Z}_2$ in more detail later. (In related matters, SU(2) and SO(3) are so-called *simple* Lie groups because they are free of nontrivial normal subgroups; in contrast, U(2) and O(3) are *not* simple Lie groups.)