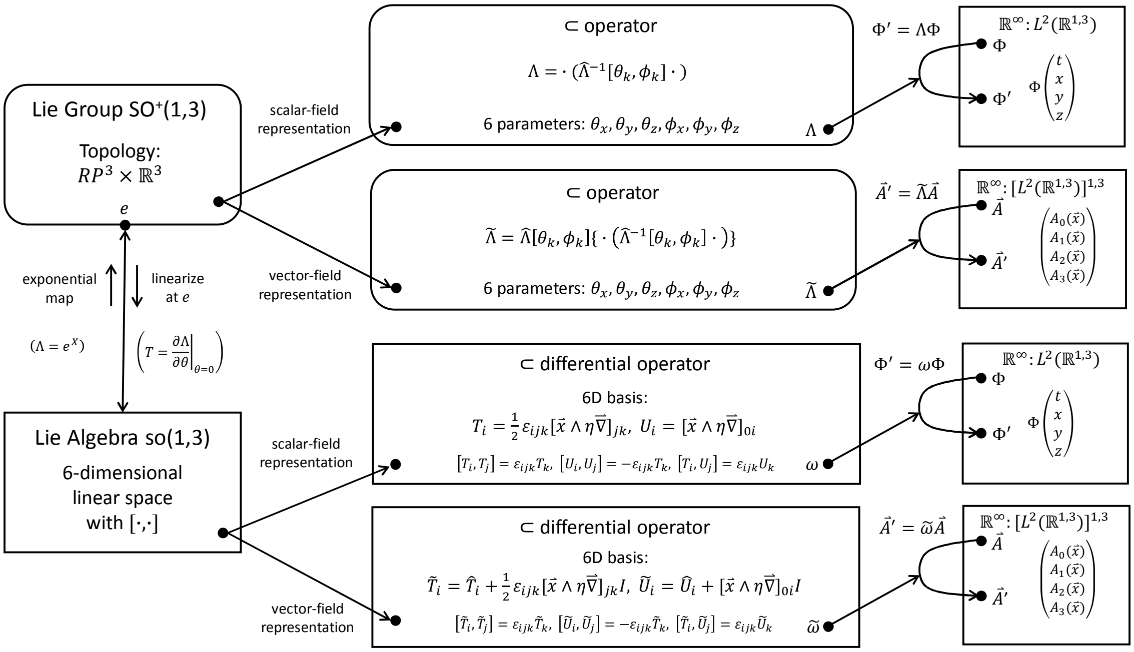
## 5.16 SO⁺(1,3): Vector-Field Representation




After the scalar field, we now turn to the *vector field* $\vec{A}(\vec{x})$, where $\vec{x} = (t, x, y, z)^T$, as before. Now, the field value is a real 4-vector, $\vec{A} = (A_0, A_1, A_2, A_3)^T$, that transforms under the defining representation of SO⁺(1,3). An example of such a vector field is the electromagnetic 4-potential. (Besides satisfying the SO⁺(1,3) space-time symmetry, the electromagnetic potential field also satisfies a gauge symmetry and thus is a *special* vector field.) The upper branch of the diagram shows again the scalar-field representation of SO⁺(1,3) for reference and the lower branch shows the vector-field representation.

How does this vector field transform under SO⁺(1,3)? From the vector-field representations of SO(3) we know that this transformation consists of two parts: a part for the 4-vector at each point in space-time and a part for the field structure, which is a function of space-time. A 4-vector by itself transforms like $\vec{A}' = \hat{\Lambda}(\theta_k, \phi_k)\vec{A}$, where $\hat{\Lambda}(\theta_k, \phi_k)$ is the defining representation of SO⁺(1,3). A scalar field in space-time transforms like $\Phi'(\vec{x}) = \Phi(\hat{\Lambda}^{-1}[\theta_k, \phi_k]\vec{x})$. Combining the two parts, we find that a 4-vector field transforms like $\vec{A}'(\vec{x}) = \hat{\Lambda}[\theta_k, \phi_k]\vec{A}(\hat{\Lambda}^{-1}[\theta_k, \phi_k]\vec{x})$. Using our informal dot notation to separate the operator from the vector field, $\vec{A}(\vec{x})$, (and suppressing the parameters) we have $\tilde{\Lambda} = \hat{\Lambda}\{\cdot\ (\hat{\Lambda}^{-1}\ \cdot)\}$, where the first dot is a placeholder for the field's name and the second dot is a placeholder for its space-time argument. This operator acts on the field like $\tilde{\Lambda}\,\vec{A}(\vec{x}) = \hat{\Lambda}\vec{A}(\hat{\Lambda}^{-1}\vec{x})$ (see the lower branch of the diagram).

As discussed earlier, the six basis generators of the scalar-field representation are $T_i = \frac{1}{2}\varepsilon_{ijk}W^{jk}$ and $U_i = W^{0i}$, where $W^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu$ and we used tensor-index notation. To bring these operators into a more familiar form, we can rewrite (using vector-matrix notation) the $W$ matrix as $W = \vec{x} \wedge \eta\vec{\nabla}$, where $\eta = \mathrm{diag}(+1, -1, -1, -1)$, $\vec{\nabla} = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)^T$, as usual, and the wedge $\wedge$ represents the *exterior product* defined by $\vec{a} \wedge \vec{b} := \vec{a}\vec{b}^T - (\vec{a}\vec{b}^T)^T$. Thus, the six basis generators can also be written as $T_i = \frac{1}{2}\varepsilon_{ijk}\left[\vec{x} \wedge \eta\vec{\nabla}\right]_{jk}$ and $U_i = \left[\vec{x} \wedge \eta\vec{\nabla}\right]_{0i}$ (see the lower branch of the diagram). For more information about the exterior product, see the Appendix “The Exterior Product; Area and Volume Elements”.

When we discussed the infinite-dimensional representations of SU(2) and SO(3), we wrote the three basis generators as $\vec{T} = -\vec{x} \times \vec{\nabla}$ (or $\vec{J} = -i\vec{x} \times \vec{\nabla}$). How is this related to the expression derived above? Instead of taking the cross product, $\vec{T} = -\vec{x} \times \vec{\nabla}$, we could have taken the exterior product resulting in the 3×3 matrix $\overline{W} = -\vec{x} \wedge \vec{\nabla}$ from which we can extract the three basis generators: $T_x = \overline{W}_{23}$, $T_y = \overline{W}_{31}$, $T_z = \overline{W}_{12}$. In fact, the 3-vector $\vec{T}$ is the Hodge dual of the 3×3 matrix $\overline{W}$ and can be written as $T_i = \frac{1}{2}\varepsilon_{ijk}\overline{W}_{jk}$. See the Appendix “The Hodge Dual in Euclidean Space” for more information.

What are the six basis generators of the vector-field representation? From our earlier discussion of 3-vector fields, we know that these generators consist of two parts: a part for the 4-vector at each point in space-time and a part for the field structure in space-time. The first part is given by the six 4×4 basis generators of the defining representation, which we now call $\hat{T}_i$ and $\hat{U}_i$. The second part is given by the six basis generators of the scalar-field representation: $T_i = \frac{1}{2}\varepsilon_{ijk}W^{jk}$ and $U_i = W^{0i}$. Combining the two parts, we get the basis generators of the vector-field representation: $\tilde{T}_i = \hat{T}_i + \frac{1}{2}\varepsilon_{ijk}W^{jk}I$ and $\tilde{U}_i = \hat{U}_i + W^{0i}I$, where $I$ is the 4×4 identity matrix, which multiplies the differential operators of the scalar-field representation to make them compatible with the matrices of the vector representation. Using vector-matrix notation, we can also write the basis generators of the vector-field representation as $\tilde{T}_i = \hat{T}_i + \frac{1}{2}\varepsilon_{ijk}\left[\vec{x} \wedge \eta\vec{\nabla}\right]_{jk}$ and $\tilde{U}_i = \hat{U}_i + \left[\vec{x} \wedge \eta\vec{\nabla}\right]_{0i}I$ (see the lower branch of the diagram). For example, the generator $\tilde{U}_z$ for a boost in the $z$ direction evaluates to

$$\tilde{U}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \left(-z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z}\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} & 0 & 0 & 1 \\ 0 & -z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & -z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} & 0 \\ 1 & 0 & 0 & -z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} \end{pmatrix}.$$

How can we express a general generator $\tilde{\omega}$ of the vector-field representation? Given the generator $\hat{\omega} = \hat{T}_i\theta_i + \hat{U}_i\phi_i$ of the 4-vector representation, we know that we can express the corresponding generator of the scalar-field representation as $\omega = \frac{1}{2}\hat{\omega}_{\mu\nu}W^{\mu\nu}$ and if we put this into the exponential map, we obtain the associated scalar-field transformation $\Lambda = \exp\left(\frac{1}{2}\hat{\omega}_{\mu\nu}W^{\mu\nu}\right)$. Similarly, we can express a general generator of the vector-field representation as $\tilde{\omega} = \hat{\omega} + \frac{1}{2}\hat{\omega}_{\mu\nu}W^{\mu\nu}I$ and if we put this into the exponential map, we obtain the associated vector-field transformation $\tilde{\Lambda} = \exp\left(\hat{\omega} + \frac{1}{2}\hat{\omega}_{\mu\nu}W^{\mu\nu}I\right)$.

Finally, we should point out that physicists like to include a factor $i$ as part of the generators and then put a $-i$ into the exponent of the exponential map to keep things consistent. Therefore, they define $M^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ from which they can pick the basis generators $J_i$ and $K_i$ in the same way we picked $T_i$ and $U_i$ from $W$ [PfS, Ch. 3.7.11]. To remain consistent, they write a scalar-field transformation as $\Lambda = \exp\left(-\frac{1}{2}i\hat{\omega}_{\mu\nu}M^{\mu\nu}\right)$.