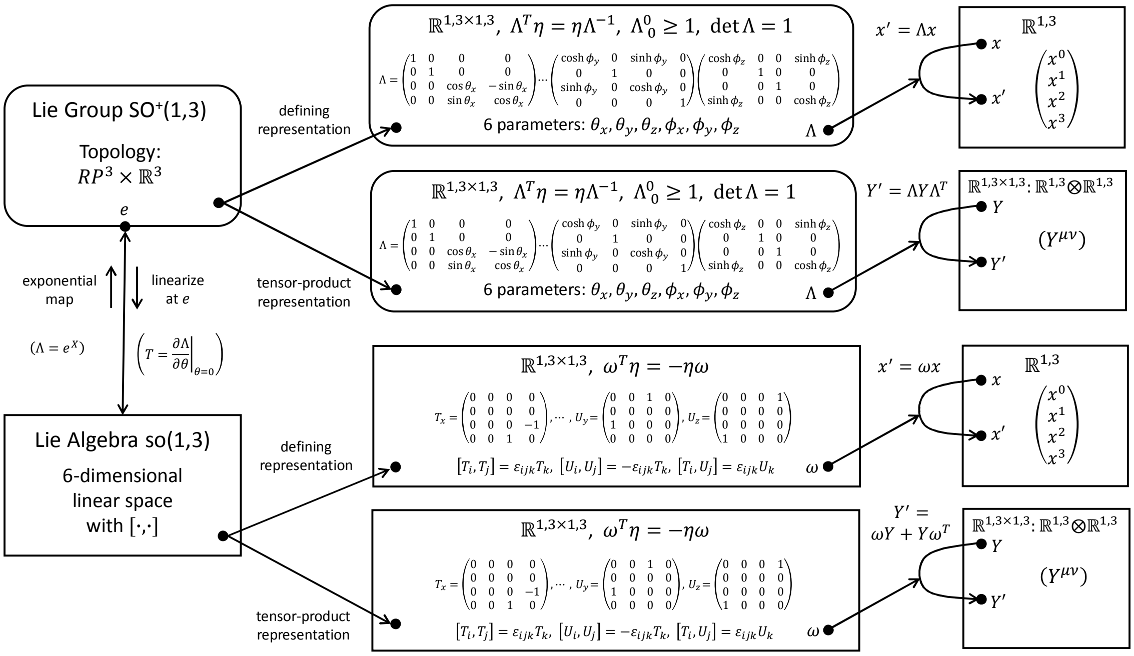
5.18 SO⁺(1,3): Tensor-Product Representations; Decomposition



Given the 4-dimensional defining representation of SO⁺(1,3), we can construct a 16-dimensional tensor-product representation on 4×4 tensors, just like we did for SO(4). But there is a twist: covectors now transform differently from vectors and hence there are four different ways to form a tensor-product space: $V\otimes V$, $V^*\otimes V$, $V\otimes V^*$, and $V^*\otimes V^*$, where V is the vector space and V^* is the dual vector space. Thus, there are four (equivalent) flavors of this tensor-product representation.

How does the Lorentz transformation Λ act on these tensors? Let's start with the $V\otimes V$ product space. As usual, we construct a prototype tensor by taking the tensor product of two vectors $Y = xy^T$; we know how the two vectors transform: $Y' = x'y'^T = (\Lambda x)(\Lambda y)^T = \Lambda xy^T\Lambda^T$ and thus we can infer how the tensor transforms: $Y' = \Lambda Y \Lambda^T$ (see the lower branch of the diagram). Using tensor-index notation, a tensor in the $V\otimes V$ product space is written with two upstairs indices and transforms like $Y'^{\mu\nu} = \Lambda^\mu_{\ \kappa} Y^{\kappa\lambda} \Lambda^\nu_{\ \lambda} = \Lambda^\mu_{\ \kappa}\Lambda^\nu_{\ \lambda} Y^{\kappa\lambda}$ [TM, Vol. 3, Ch. 6.2.1].

For the other three product spaces, we follow the same line of thought. For example, for the $V^*\otimes V^*$ product space, we construct a prototype from two covectors $Y = \bar{x}\bar{y}^T$; we know how the two covectors transform: $Y' = \bar{x}'\bar{y}'^T = (\Lambda^{-1T}\bar{x})(\Lambda^{-1T}\bar{y})^T = \Lambda^{-1T}\bar{x}\bar{y}^T\Lambda^{-1}$ and thus we can infer how this tensor transforms: $Y' = \Lambda^{-1T} Y \Lambda^{-1}$. Using tensor-index notation, we write $Y'_{\mu\nu} = \Lambda_\mu^{\ \kappa} Y_{\kappa\lambda} \Lambda_\nu^{\ \lambda} = \Lambda_\mu^{\ \kappa}\Lambda_\nu^{\ \lambda} Y_{\kappa\lambda}$.

From our discussion of SO(4), we know that its 16-dimensional tensor-product representation is reducible into a 10-dimensional *symmetric* and a 6-dimensional *antisymmetric* representation. This is also true for the general linear group GL(4) and thus for any of its subgroups, including SO⁺(1,3).

An example of a physical quantity that transforms under the *antisymmetric* tensor representation of SO⁺(1,3) is the *electromagnetic field-strength tensor* (a.k.a. *Faraday tensor*) $F^{\mu\nu}$. This tensor and its dual, $F_{\mu\nu}$, can be written as

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad (F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

where E_x, E_y, E_z are the components of the *electric field* vector $\vec{E}$ and B_x, B_y, B_z are the components of the *magnetic field* vector $\vec{B}$; as usual, we chose units in which $c = 1$ [QFTGA, Ch. 5.2].

An example of a physical quantity that transforms under the *symmetric* tensor representation of SO⁺(1,3) is the *energy-momentum (density) tensor* (a.k.a. *stress-energy tensor*) $T^{\mu\nu}$. For an electromagnetic field, we have $T^{\mu\nu} = F^{\mu\sigma}F_{\sigma}^{\nu} - \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau}$ [TM, Vol 3, Ch. 11.5.3]. This tensor and its dual, $T_{\mu\nu}$, can be written as

$$(T^{\mu\nu}) = \begin{pmatrix} \rho & S_x & S_y & S_z \\ S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y & -\sigma_{xy} & -\sigma_{yy} & -\sigma_{yz} \\ S_z & -\sigma_{xz} & -\sigma_{yz} & -\sigma_{zz} \end{pmatrix}, \qquad (T_{\mu\nu}) = \begin{pmatrix} \rho & -S_x & -S_y & -S_z \\ -S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ -S_y & -\sigma_{xy} & -\sigma_{yy} & -\sigma_{yz} \\ -S_z & -\sigma_{xz} & -\sigma_{yz} & -\sigma_{zz} \end{pmatrix},$$

where $\rho = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)$ is the *energy density*, S_x, S_y, S_z are the components of the *Poynting vector*, $\vec{S} = \vec{E} \times \vec{B}$, quantifying the *momentum density*, σ_{ii} are the *normal stress* components, and σ_{ij} for $i \neq j$ are the *shear stress* components of the stress tensor $\sigma = \vec{E}\vec{E}^T + \vec{B}\vec{B}^T - \rho I$ [TM, Vol. 3, Ch. 11.5.3].

Just like for SO(4), the 10-dimensional symmetric tensor representation of SO⁺(1,3) breaks up into a 9-dimensional traceless symmetric representation and a 1-dimensional (trivial) representation for the trace. Note that for SO⁺(1,3), the trace is given by $Y_{\mu}^{\mu} = \eta_{\mu\nu}Y^{\mu\nu} = Y^{00} - Y^{11} - Y^{22} - Y^{33}$, not the simple diagonal sum we are used to from the orthogonal case. For the above electromagnetic energy-momentum tensor, the trace $\rho + \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ evaluates to zero: $\frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) + |\vec{E}|^2 + |\vec{B}|^2 - \frac{3}{2}(|\vec{E}|^2 + |\vec{B}|^2) = 0$.

Just like for SO(4), the antisymmetric tensor representation of SO⁺(1,3) is equivalent to the adjoint representation, both of which are six dimensional. However, because the action on the tensor-product space, $Y' = \Lambda Y \Lambda^T$, and the action on the space of generators (which are not necessarily antisymmetric), $Y' = \Lambda Y \Lambda^{-1}$, are different for SO⁺(1,3), this is not as obvious as for the orthogonal case.

Finally, just like for SO(4), the 6-dimensional antisymmetric tensor representation of SO⁺(1,3) breaks up into two 3-dimensional representations, known again as the *self-dual* and *anti-self-dual* representations. This will be the subject of the next example.

In summary, the full decomposition of the tensor-product representation of SO⁺(1,3) into irreducibles is $\mathbf{4}\otimes\mathbf{4} = \mathbf{9}\oplus\mathbf{3}\oplus\overline{\mathbf{3}}\oplus\mathbf{1}$. In terms of irreducible representations, SO⁺(1,3) looks just like SO(4). Indeed, we will see later that the finite-dimensional representations of the two groups can be classified and labeled in the same way!