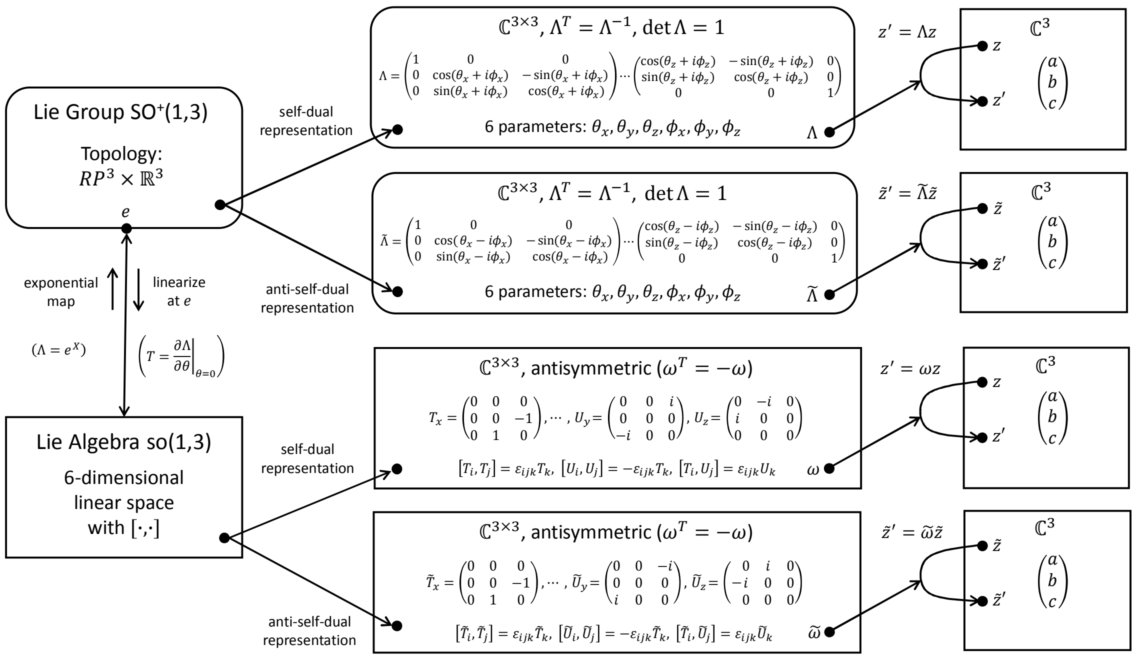
5.20 SO⁺(1,3): Self-Dual and Anti-Self-Dual Representations on 3D Vectors



In the previous example we found two 3-dimensional representations of SO⁺(1,3), one acting on self-dual and one acting on anti-self-dual tensors. It is instructive to rewrite these representations in a form where they act on 3-component column vectors. For example, letting the matrix for rotation about the x axis (from the previous example) act on a general self-dual matrix, $Y^+ \to \Lambda Y^+ \Lambda^T$, yields

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & -ic & ib \\ -b & ic & 0 & -ia \\ -c & -ib & ia & 0 \end{pmatrix} \to \begin{pmatrix} 0 & a & b\cos\theta_x - c\sin\theta_x & b\sin\theta_x + c\cos\theta_x \\ -a & 0 & -i(b\sin\theta_x + c\cos\theta_x) & i(b\cos\theta_x - c\sin\theta_x) \\ -(b\cos\theta_x - c\sin\theta_x) & i(b\sin\theta_x + c\cos\theta_x) & 0 & -ia \\ -(b\sin\theta_x + c\cos\theta_x) & -i(b\cos\theta_x - c\sin\theta_x) & ia & 0 \end{pmatrix},$$

which is again a self-dual matrix, as expected. Now, focusing on the three free parameters a, b, and c, the same transformation can be written in the simpler (unpacked) form

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a \\ b\cos\theta_x - c\sin\theta_x \\ b\sin\theta_x + c\cos\theta_x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = R_{yz} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Repeating this procedure for the remaining five factors yields the overall 3×3 transformation matrix $\Lambda' = R_{yz}(\theta_x) \cdot R_{zx}(\theta_y) \cdot R_{xy}(\theta_z) \cdot B_{tx}(\phi_x) \cdot B_{ty}(\phi_y) \cdot B_{tz}(\phi_z)$, where

$$R_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix},\ R_{zx} = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix},\ R_{xy} = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_{tx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\phi_x & -i\sinh\phi_x \\ 0 & i\sinh\phi_x & \cosh\phi_x \end{pmatrix},\ B_{ty} = \begin{pmatrix} \cosh\phi_y & 0 & i\sinh\phi_y \\ 0 & 1 & 0 \\ -i\sinh\phi_y & 0 & \cosh\phi_y \end{pmatrix},\ B_{tz} = \begin{pmatrix} \cosh\phi_z & -i\sinh\phi_z & 0 \\ i\sinh\phi_z & \cosh\phi_z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rotation matrices R_{ij} are real and thus rotate the real and imaginary parts of the complex 3D vector independently, whereas the boost matrices B_{ti} are complex and mix the real and imaginary parts of the vector together. In the previous example we saw that the complex electromagnetic field $\vec{E} + i\vec{B}$ transforms under the self-dual representation. Now, we see that rotations act on $\vec{E}$ and $\vec{B}$ independently, whereas boosts mix $\vec{E}$ and $\vec{B}$ together, which makes perfect physical sense!

We can simplify the expression for the overall transformation by rearranging the factors like $\Lambda = R_{yz}(\theta_x)B_{tx}(\phi_x) \cdot R_{zx}(\theta_y)B_{ty}(\phi_y) \cdot R_{xy}(\theta_z)B_{tz}(\phi_z)$ and using the identities $\sin(i\phi) = i\sinh(\phi)$ and $\cos(i\phi) = \cosh(\phi)$ to combine adjacent matrices (see the upper branch of the diagram):

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x + i\phi_x) & -\sin(\theta_x + i\phi_x) \\ 0 & \sin(\theta_x + i\phi_x) & \cos(\theta_x + i\phi_x) \end{pmatrix} \begin{pmatrix} \cos(\theta_y + i\phi_y) & 0 & \sin(\theta_y + i\phi_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y + i\phi_y) & 0 & \cos(\theta_y + i\phi_y) \end{pmatrix} \begin{pmatrix} \cos(\theta_z + i\phi_z) & -\sin(\theta_z + i\phi_z) & 0 \\ \sin(\theta_z + i\phi_z) & \cos(\theta_z + i\phi_z) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Although this matrix differs from the original one, it is the same representation parametrized in a different way.) Amazingly, this simplified transformation is just 3D rotation with *complex angles*! The imaginary parts of the complex angles are the rapidities. The matrix Λ is a *complex* matrix that satisfies $\Lambda^T = \Lambda^{-1}$ and $\det\Lambda = 1$, that is, it is *complex orthogonal*. In other words, this representation of SO⁺(1,3) is just the *complexification* (= analytic continuation) of the defining representation of SO(3)!

Next, let's unpack the generators. In fact, we already did this for U_z in the previous example:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} -ib \\ ia \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = U_z \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Repeating this procedure for the remaining five basis generators, we find that the T_k equal the basis generators of the defining representation of so(3) and the U_k equal iT_k. Thus, a general generator ω is a *complex antisymmetric* matrix: $\omega^T = -\omega$. In other words, this representation of so(1,3) is the complexification of the defining representation of so(3). This fact can be expressed as $\mathrm{so}(1,3) = \mathrm{so}(3)_{\mathbb{C}}$. It is rather magical how the concepts of relativistic boosts and time arise from the analytic continuation of 3D space rotations!

Applying the above unpacking procedure to the anti-self-dual representation yields the 3×3 transformation matrix $\widetilde{\Lambda}$ shown in the lower branch of the diagram, which now depends on the parameter combination $\theta_k - i\phi_k$ instead of $\theta_k + i\phi_k$. In fact, the anti-self-dual representation is the *complex conjugate* of the self-dual representation: $\widetilde{\Lambda} = \Lambda^*$. Similarly, the basis generators of the anti-self-dual representation are $\widetilde{T}_k = T_k$ and $\widetilde{U}_k = -iT_k$ and thus $\widetilde{\omega} = \omega^*$. Like for SO(4), the self-dual and anti-self-dual representations of SO⁺(1,3) are inequivalent, that is, they are *not* related by a similarity transformation. (Since a similarity transformation cannot change the eigenvalues of a matrix, it is, in general, not possible to map all the complex eigenvalues to their *complex conjugate* [QTGR, Ch. 41.1].)

Comparing the generators of the self-dual and anti-self-dual representations of SO⁺(1,3), we see that the generators of rotation are the same, $\widetilde{T}_k = T_k$, while the generators of boost have opposite signs, $\widetilde{U}_k = -U_k$. This is the tell-tale sign of *parity inversion*! Here we have an example of two *inequivalent* representations that are related by parity inversion. If the self-dual representation is considered left-handed, then the anti-self-dual representation is right-handed. In contrast, the defining representation of SO⁺(1,3) and its parity-inverted version were equivalent, as we saw earlier.