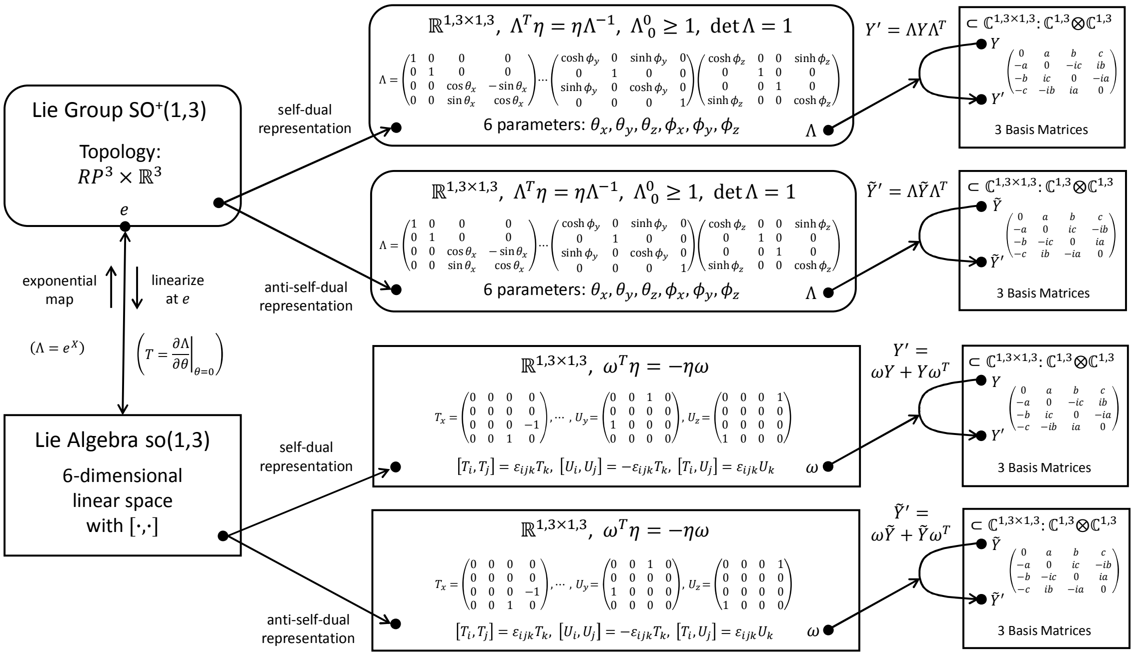
5.19 SO⁺(1,3): Self-Dual and Anti-Self-Dual Tensor Representations



Before we can construct the self-dual and anti-self-dual tensor representations of SO⁺(1,3), we need to know how to calculate the Hodge dual in Minkowski space. Given the antisymmetric rank-2 tensor $Y^{\mu\nu}$, the Hodge dual in Minkowski space is the antisymmetric rank-2 tensor $\hat{Y}^{\rho\sigma} = \frac{1}{2}\eta^{\rho\kappa}\eta^{\sigma\lambda}\varepsilon_{\kappa\lambda\mu\nu}Y^{\mu\nu}$ [RtR, Ch. 19.2; GFKG, Ch. I.5, p. 91]. We can think of this map as a two-step process: first, we apply the formula for the Euclidean case, $\frac{1}{2}\varepsilon_{\kappa\lambda\mu\nu}Y^{\mu\nu}$; then, we use the metric tensor to pull the lower indices back up. As usual, letting $\eta \to I$ takes us back to the Euclidean case.

Evaluating $\hat{Y}^{\rho\sigma} = \frac{1}{2}\eta^{\rho\kappa}\eta^{\sigma\lambda}\varepsilon_{\kappa\lambda\mu\nu}Y^{\mu\nu}$ for the general antisymmetric 4×4 matrix shown on the left-hand side results in the dual matrix shown to the right-hand side:

$$Y = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \nu & \mu \\ -\beta & -\nu & 0 & \lambda \\ -\gamma & -\mu & -\lambda & 0 \end{pmatrix} \to \hat{Y} = \begin{pmatrix} 0 & -\lambda & \mu & -\nu \\ \lambda & 0 & \gamma & -\beta \\ -\mu & -\gamma & 0 & \alpha \\ \nu & \beta & -\alpha & 0 \end{pmatrix}.$$

The dual matrix is again antisymmetric, but its components got moved around. If we take the Hodge dual for a second time, we *don't* get back to the original matrix, as we did in the Euclidean case, instead we get *minus* the original matrix. Because of this sign difference, the definition of *self-dual* and *anti-self-dual* matrices in Minkowski space is somewhat different from that in Euclidean space. Now, the self-dual matrix has the property that its dual is i times the original: $\hat{Y}^+ = iY^+$ (whereas we had $\hat{Y}^+ = Y^+$ in the Euclidean case). Similarly, the anti-self-dual matrix has the property that its dual is $-i$ times the original: $\hat{Y}^- = -iY^-$ (whereas we had $\hat{Y}^- = -Y^-$ in the Euclidean case). The reason for these differences is that the eigenvalues of the Hodge-star operator are now $\pm i$, whereas they were ± 1 for the Euclidean case [GFKG Ch. I.5, p. 97]. As usual, the Minkowski metric makes everything a little bit messier!

Given any antisymmetric matrix Y, we can decompose it like $Y = Y^+ + Y^-$, where $Y^+ = \frac{1}{2}(Y + \hat{Y}/i)$ is self-dual and $Y^- = \frac{1}{2}(Y - \hat{Y}/i)$ is anti-self-dual. To confirm these properties, we calculating the Hodge duals $\hat{Y}^+ = \frac{1}{2}(\hat{Y} - Y/i) = i\,\frac{1}{2}(Y + \hat{Y}/i) = iY^+$ and $\hat{Y}^- = \frac{1}{2}(\hat{Y} + Y/i) = -i\,\frac{1}{2}(Y - \hat{Y}/i) = -iY^-$. Using the component names introduced above, we find

$$Y^+ = \frac{1}{2}\begin{pmatrix} 0 & \alpha + i\lambda & \beta - i\mu & \gamma + i\nu \\ -\alpha - i\lambda & 0 & \nu - i\gamma & \mu + i\beta \\ -\beta + i\mu & -\nu + i\gamma & 0 & \lambda - i\alpha \\ -\gamma - i\nu & -\mu - i\beta & -\lambda + i\alpha & 0 \end{pmatrix}, \quad Y^- = \frac{1}{2}\begin{pmatrix} 0 & \alpha - i\lambda & \beta + i\mu & \gamma - i\nu \\ -\alpha + i\lambda & 0 & \nu + i\gamma & \mu - i\beta \\ -\beta - i\mu & -\nu - i\gamma & 0 & \lambda + i\alpha \\ -\gamma + i\nu & -\mu + i\beta & -\lambda - i\alpha & 0 \end{pmatrix}.$$

The matrices are now complex, but again, each one depends on only *three* parameters:

$$Y^+ = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -ic & ib \\ -b & ic & 0 & -ia \\ -c & -ib & ia & 0 \end{pmatrix}, \quad Y^- = \begin{pmatrix} 0 & a^* & b^* & c^* \\ -a^* & 0 & ic^* & -ib^* \\ -b^* & -ic^* & 0 & ia^* \\ -c^* & ib^* & -ia^* & 0 \end{pmatrix}.$$

The diagram shows the self-dual and anti-self-dual representations of SO$^+$(1,3), which are both three dimensional. In the upper branch, the Lorentz transformations act on self-dual matrices (named Y instead of Y^+ to avoid clutter), always producing another self-dual matrix and in the lower branch they act on anti-self-dual matrices (named $\tilde{Y}$), always producing another anti-self-dual matrix.

Let's try this out by acting with the generator U_z on the general self-dual matrix Y^+, that is, we calculate $U_z Y^+ + Y^+ U_z^T$:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -ic & ib \\ -b & ic & 0 & -ia \\ -c & -ib & ia & 0 \end{pmatrix} + \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -ic & ib \\ -b & ic & 0 & -ia \\ -c & -ib & ia & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ib & ia & 0 \\ ib & 0 & 0 & -a \\ -ia & 0 & 0 & -b \\ 0 & a & b & 0 \end{pmatrix}.$$

Indeed, the result is again a self-dual matrix: the three independent parameters (a, b, c) in the self-dual matrix get mapped to $(-ib, ia, 0)$. Repeating this procedure for the remaining five generators confirms that the Y^+ do furnish a separate representation. The same is true for the anti-self-dual matrices Y^-.

Taking the Hodge dual of the electromagnetic field-strength tensor, $F^{\mu\nu}$, introduced earlier, exchanges the electric and magnetic fields and flips the sign of one of them, $\vec{E} \to -\vec{B}$ and $\vec{B} \to \vec{E}$:

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \to (\hat{F}^{\mu\nu}) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

The self-dual and anti-self-dual field tensors, $F^+ = \frac{1}{2}(F + \hat{F}/i)$ and $F^- = \frac{1}{2}(F - \hat{F}/i)$, evaluate to

$$F^+ = \frac{1}{2}\begin{pmatrix} 0 & -E_x - iB_x & -E_y - iB_y & -E_z - iB_z \\ E_x + iB_x & 0 & -B_z + iE_z & B_y - iE_y \\ E_y + iB_y & B_z - iE_z & 0 & -B_x + iE_x \\ E_z + iB_z & -B_y + iE_y & B_x - iE_x & 0 \end{pmatrix}, \quad F^- = \frac{1}{2}\begin{pmatrix} 0 & -E_x + iB_x & -E_y + iB_y & -E_z + iB_z \\ E_x - iB_x & 0 & -B_z - iE_z & B_y + iE_y \\ E_y - iB_y & B_z + iE_z & 0 & -B_x - iE_x \\ E_z - iB_z & -B_y - iE_y & B_x + iE_x & 0 \end{pmatrix}.$$

Thus, the complex fields $\vec{E} + i\vec{B}$ and $\vec{E} - i\vec{B}$ transform under the self-dual and anti-self-dual representations of the Lorentz group, respectively!