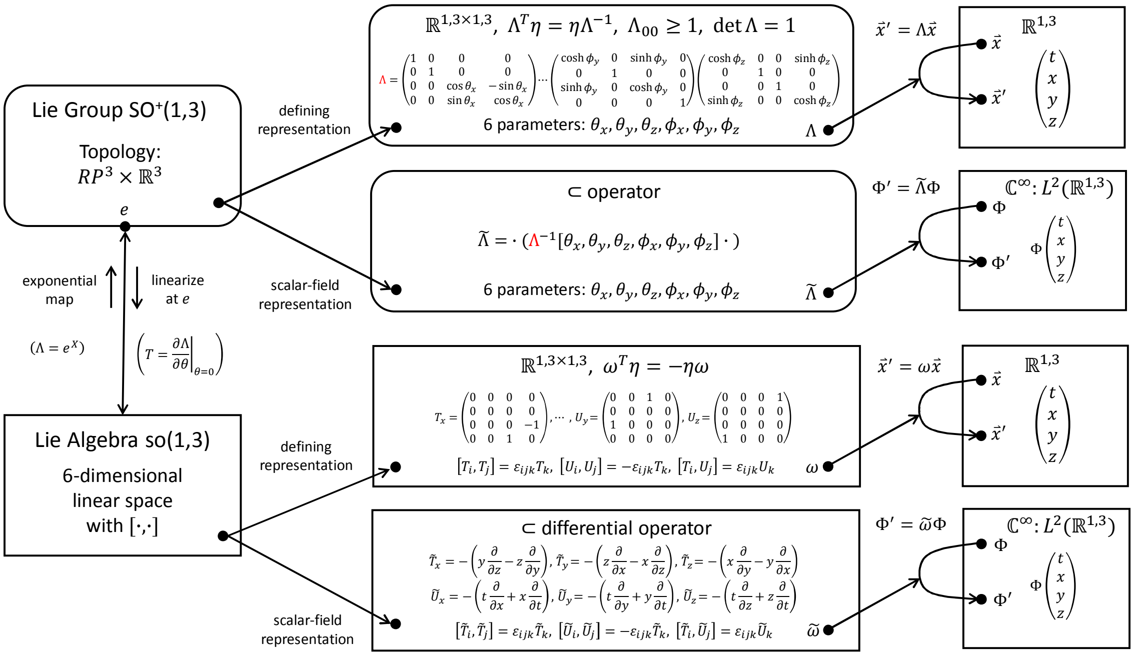
5.14 SO⁺(1,3): Scalar-Field Representation



A scalar function of space time, $\Phi(\vec{x})$, where $\vec{x} = (t, x, y, z)^T$, is known as a *scalar field*. (Note that we marked the 4-vector with a harpoon rather than the arrow used for 3-vectors.) Scalar fields that respect the laws of special relativity furnish an infinite-dimensional representation of SO⁺(1,3). Here, we choose the square-integrable complex scalar fields, which are elements of $\mathbb{C}^\infty$ and $L^2(\mathbb{R}^{1,3})$. The upper branch of the diagram shows again the defining representation of SO⁺(1,3) for reference and the lower branch shows the infinite-dimensional representation on complex scalar fields.

How does SO⁺(1,3) act on a scalar field? Analogous to what we did for SU(2) and SO(3), we use the inverse of the defining representation, Λ^{-1}, to transform the 4D argument $\vec{x}$ of our field $\Phi(\vec{x})$, that is, $\Phi'(\vec{x}) = \Phi(\Lambda^{-1}[\theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z]\ \vec{x})$. To write the field transformation as a disembodied operator we use the informal dot notation $\tilde{\Lambda} = \cdot\,(\Lambda^{-1}\cdot)$, where the first dot is a placeholder for the field's name and the second dot is a placeholder for its space-time argument. This operator acts on the field like $\tilde{\Lambda}\,\Phi(\vec{x}) = \Phi(\Lambda^{-1}\vec{x})$ (see the lower branch of the diagram).

To find the elements of the Lie algebra, we differentiate the group elements while they act on the field and then split the result into a differential operator and the field it acts on. For example, for a rotation about the x axis, which depends on the angle θ_x, the corresponding field transformation is

$$\tilde{\Lambda}(\theta_x)\ \Phi(\vec{x}) = \Phi(\Lambda^{-1}[\theta_x]\ \vec{x}) = \Phi(\Lambda[-\theta_x]\ \vec{x}) = \Phi\begin{pmatrix} t \\ x \\ y\cos\theta_x + z\sin\theta_x \\ -y\sin\theta_x + z\cos\theta_x \end{pmatrix}.$$

Taking the derivative with respect to θ_x (using the chain rule) and setting θ_x to zero yields a differential operator acting on the field, which, when split off, is the basis generator $\tilde{T}_x$:

$$\tilde{T}_x\Phi(\vec{x}) = \frac{\partial\Phi}{\partial y}\cdot z + \frac{\partial\Phi}{\partial z}\cdot(-y), \qquad \tilde{T}_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

The basis generators for rotation about the y and z axes are of the same form. In fact, the three basis generators are essentially the same ones we obtained earlier for the field representations of SU(2) or SO(3). Note that here we don't multiply the basis generators by i and call them $\tilde{T}_i$ instead of $\tilde{J}_i$.

Doing the same thing for a boost in the positive z direction, which depends on the rapidity parameter ϕ_z, we find the field transformation

$$\tilde{\Lambda}(\phi_z)\,\Phi(\vec{x}) = \Phi(\Lambda^{-1}[\phi_z]\,\vec{x}) = \Phi(\Lambda[-\phi_z]\,\vec{x}) = \Phi\begin{pmatrix} t\cosh\phi_z - z\sinh\phi_z \\ x \\ y \\ -t\sinh\phi_z + z\cosh\phi_z \end{pmatrix}.$$

Taking the derivative with respect to ϕ_z and setting ϕ_z to zero yields a differential operator acting on the field, which, when split off, is the basis generator $\tilde{U}_z$:

$$\tilde{U}_z\Phi(\vec{x}) = \frac{\partial\Phi}{\partial t}\cdot(-z) + \frac{\partial\Phi}{\partial z}\cdot(-t), \qquad \tilde{U}_z = -z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z}.$$

The basis generators for boosts in the x and y directions are of the same form. The lower branch of the diagram shows the explicit form of all six basis generators.

Given a general generator ω of the defining representation, the corresponding generator acting on the scalar field is $\tilde{\omega}\,\Phi(\vec{x}) = [\vec{\nabla}\Phi(\vec{x})]^T(-\omega\vec{x})$, which, when split off, becomes $\tilde{\omega} = -(\omega\vec{x})^T\vec{\nabla}$, where $\vec{\nabla} = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)^T$. Rewritten in tensor-index notation, we have $\tilde{\omega} = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}$, which represents a "scalar product" between the matrices $\omega^{\mu}{}_{\nu}$ and $-x^{\nu}\partial_{\mu}$. We can massage this expression, such that the components of the second matrix become the *basis generators* of the infinite-dimensional representation. First, we lower the first index of the generator $\omega^{\mu}{}_{\nu}$ and pull up the corresponding index of the second matrix: $\tilde{\omega} = -\eta_{\mu\sigma}\omega^{\sigma}{}_{\nu}x^{\nu}\eta^{\mu\sigma}\partial_{\sigma} = -\omega_{\mu\nu}x^{\nu}\partial^{\mu}$. This move makes $\omega_{\mu\nu}$ antisymmetric:

$$\omega = (\omega^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & 0 & -\theta_z & \theta_y \\ \phi_y & \theta_z & 0 & -\theta_x \\ \phi_z & -\theta_y & \theta_x & 0 \end{pmatrix} \;\rightarrow\; \eta\omega = (\omega_{\mu\nu}) = \begin{pmatrix} 0 & \phi_x & \phi_y & \phi_z \\ -\phi_x & 0 & \theta_z & -\theta_y \\ -\phi_y & -\theta_z & 0 & \theta_x \\ -\phi_z & \theta_y & -\theta_x & 0 \end{pmatrix}$$

Then, we antisymmetrize the second matrix, which we are allowed to do because the first matrix is antisymmetric: $-x^{\nu}\partial^{\mu} \rightarrow \frac{1}{2}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$. Thus, we finally arrive at $\tilde{\omega} = \frac{1}{2}\omega_{\mu\nu}W^{\mu\nu}$, where

$$W = (W^{\mu\nu}) = (x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) = \begin{pmatrix} 0 & -t\frac{\partial}{\partial x} - x\frac{\partial}{\partial t} & -t\frac{\partial}{\partial y} - y\frac{\partial}{\partial t} & -t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t} \\ t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} & 0 & -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} & -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} \\ t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} & x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} & 0 & -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y} \\ t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} & x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} & y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} & 0 \end{pmatrix}.$$

From this matrix we can extract the six basis generators, as desired: $\tilde{T}_x = W^{23}$, $\tilde{T}_y = W^{31}$, $\tilde{T}_z = W^{12}$, $\tilde{U}_x = W^{01}$, $\tilde{U}_y = W^{02}$, $\tilde{U}_z = W^{03}$. More compactly, we can write $\tilde{T}_i = \frac{1}{2}\varepsilon_{ijk}W^{jk}$ and $\tilde{U}_i = W^{0i}$.