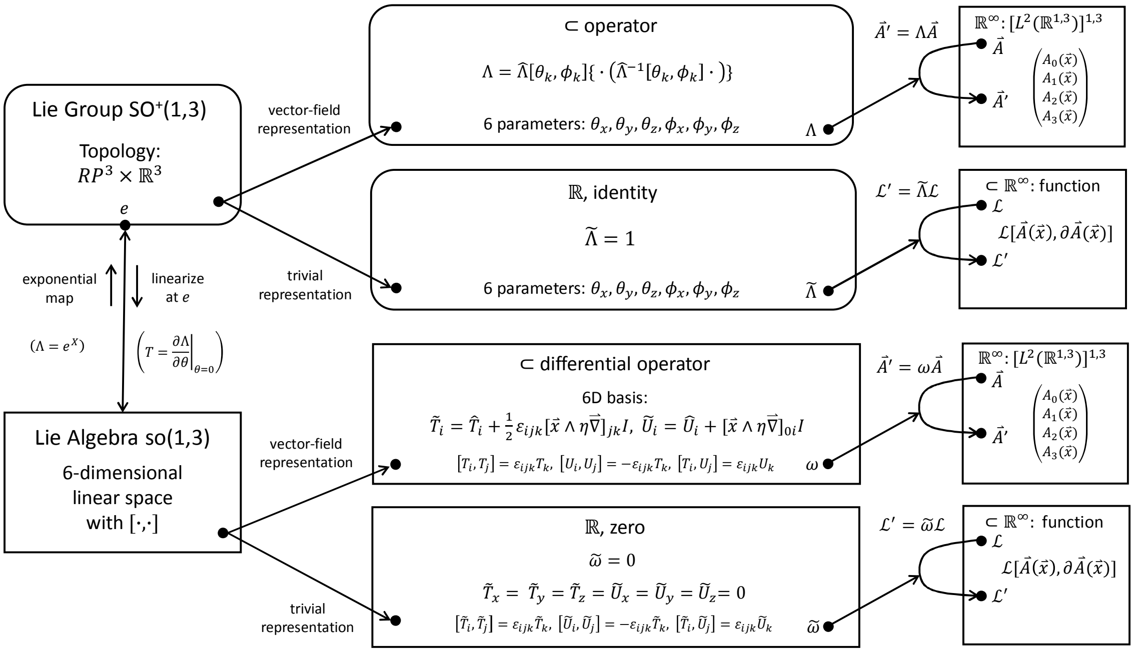
5.17 SO⁺(1,3): Application to Vector-Field Dynamics; Proca and Maxwell Lagrangians



How does a free vector field evolve in time? As we discussed for the scalar field, to respect the laws of special relativity, the associated action must remain invariant under Lorentz transformations. In other words, whereas the vector field furnishes an infinite-dimensional representation of SO⁺(1,3), the action functional must furnish the *trivial representation* of SO⁺(1,3). The upper branch of the diagram shows again the vector-field representation of SO⁺(1,3) and the lower branch shows the trivial representation acting on the Lagrangian density, which is a real function of the field, $\vec{A}(\vec{x})$, and its derivatives, $\partial\vec{A}(\vec{x})$.

What candidate terms for the Lagrangian density can we construct from the vector field and its derivatives? Like for the scalar field, we consider only terms with *first-order derivatives* and field products up to *second order* [PfS, Ch. 4.2].

From the vector field itself, we can construct the square magnitude using the Minkowski metric, which is invariant and satisfies the above constraints. This invariant is $\eta_{\mu\nu}A^\nu(\vec{x})A^\mu(\vec{x}) = A_\mu(\vec{x})A^\mu(\vec{x})$, where we used tensor-index notation [PfS, Ch. 6.4].

Taking the derivatives of the vector field, $\partial_\mu A^\nu(\vec{x})$, and contracting the indices of the result yields the divergence of the vector field, $\partial_\mu A^\mu(\vec{x})$. This is an invariant that satisfies our constraints, but it turns out that the action based on this invariant has no effect on the equations of motion and therefore can be ignored [PfS, Ch. 6.4]. Taking the derivatives of the vector field without contracting the indices yields the tensor field $\partial_\mu A^\nu(\vec{x})$. By multiplying this field with its index-raised/lowered cousin, $\partial^\kappa A_\lambda(\vec{x})$, and contracting two pairs of indices we can construct invariants. In fact, there are two ways to do the contractions, namely $\partial_\mu A_\nu \partial^\mu A^\nu$ and $\partial_\nu A_\mu \partial^\mu A^\nu$, both of which are valid candidate terms for the Lagrangian density [PfS, Ch. 6.4].

Linearly combining these candidate terms yields the general Lagrangian density for a 4-vector field transforming under the trivial representation of $SO^+(1,3)$ and satisfying our additional constraints: $a\partial_\mu A_\nu(\vec{x})\partial^\mu A^\nu(\vec{x}) + b\partial_\nu A_\mu(\vec{x})\partial^\mu A^\nu(\vec{x}) + cA_\mu(\vec{x})A^\mu(\vec{x})$, where a, b and c are real constants.

In physics, we are not so much interested in general 4-vector fields than in *gauge fields*. Gauge fields are special 4-vector fields that define how to "parallel transport" the values of other fields. In other words, they are *connections* that appear in *covariant derivatives*. We will discuss these concepts in detail when we come to gauge theory. As we will see, for $A_\mu(\vec{x})$ to be a U(1) gauge field, the Lagrangian density must remain invariant under the transformation $A'_\mu(\vec{x}) = A_\mu(\vec{x}) + \partial_\mu\alpha(\vec{x})$, where $\alpha(\vec{x})$ is an arbitrary smooth function. It turns out that this additional symmetry constraint forces $b = -a$ [PfS, Ch. 6.4, Ch. 7.1.2]. It is conventional to call $c/a = -m^2$ and choose $a = -1/2$. After these substitutions, we arrive at the (real) Proca Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu A_\nu(\vec{x})\partial^\mu A^\nu(\vec{x}) - \partial_\nu A_\mu(\vec{x})\partial^\mu A^\nu(\vec{x}) - m^2 A_\mu(\vec{x})A^\mu(\vec{x})\right].$$

The first two terms are often rewritten as a product, leading to the following form of the (real) Proca Lagrangian density [QFTGA, Ch. 13.2]:

$$\mathcal{L} = -\frac{1}{4}\left[\partial_\mu A_\nu(\vec{x}) - \partial_\nu A_\mu(\vec{x})\right]\left[\partial^\mu A^\nu(\vec{x}) - \partial^\nu A^\mu(\vec{x})\right] + \frac{1}{2}m^2 A_\mu(\vec{x})A^\mu(\vec{x}).$$

The product of the two square-bracket expressions is often abbreviated as $F_{\mu\nu}F^{\mu\nu}$. Finally, if we set $m = 0$, another consequence of the above-mentioned gauge-symmetry constraint, we arrive at the free Maxwell Lagrangian density (which describes a proper U(1) gauge field):

$$\mathcal{L} = -\frac{1}{4}\left[\partial_\mu A_\nu(\vec{x}) - \partial_\nu A_\mu(\vec{x})\right]\left[\partial^\mu A^\nu(\vec{x}) - \partial^\nu A^\mu(\vec{x})\right].$$

For the equations of motion that follow from these Lagrangian densities as well as the solutions of those equations, see the Appendix "Decoding the Proca and Free Maxwell Equations".