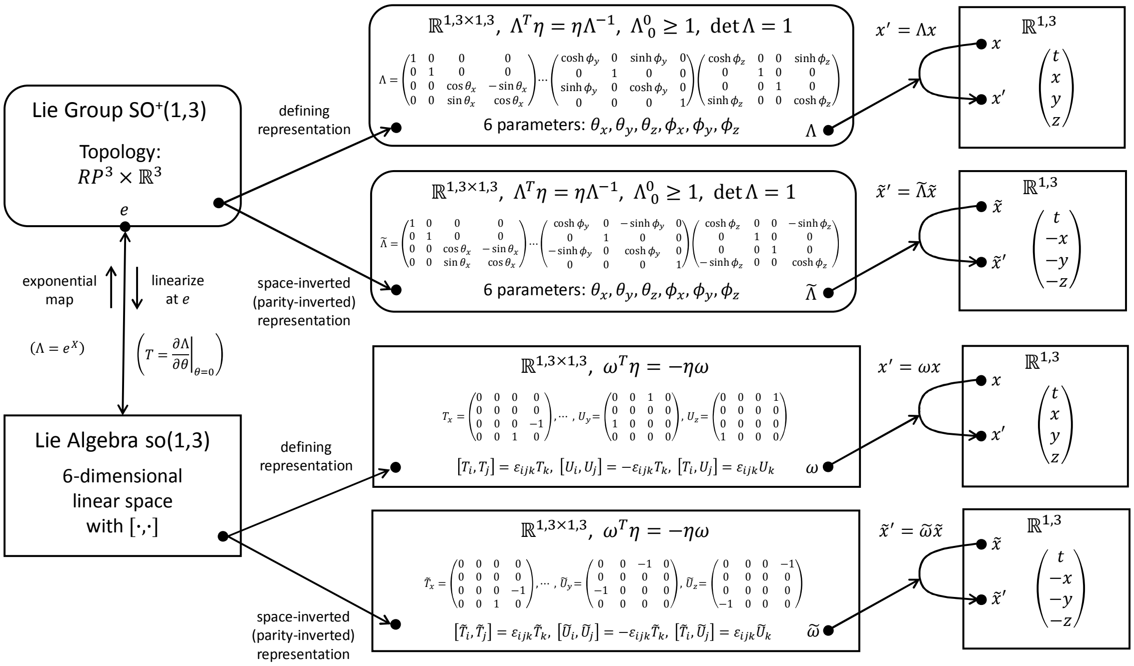
5.13 SO⁺(1,3): Space-Inverted and Time-Reversed Representations; Parity



We can *space invert* or *parity invert* a 4-vector by changing the signs of all three position coordinates. In other words, given the 4-vector x, the parity-inverted 4-vector is Px, where [QFTGA, Ch. 15.4]

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can visualize parity inversion as a mirror reflection in the xy plane (which inverts the z coordinate) followed by a 180° rotation about the z axis (which inverts the x and y coordinates). Parity inversion changes the orientation of the basis of the vector space from right handed to left handed (or vice versa).

To find the action of the group elements on parity-inverted 4-vectors, $\tilde{x}' = \tilde{\Lambda}\tilde{x}$, we substitute $\tilde{x} = Px$ and $\tilde{x}' = Px'$, which results in $Px' = \tilde{\Lambda}Px$. Moving P to the right-hand side and using that $P^{-1} = P$ yields $x' = P\tilde{\Lambda}Px$. Comparing this to $x' = \Lambda x$, we conclude that $\Lambda = P\tilde{\Lambda}P$ or, after inversion, $\tilde{\Lambda} = P\Lambda P$.

What happens to the matrix components of Λ when we switch from a given representation to its parity-inverted version? Evaluating $P\Lambda P$ for a general matrix Λ reveals

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b & -c & -d \\ -e & f & g & h \\ -i & j & k & l \\ -m & n & o & p \end{pmatrix},$$

that is, parity inversion flips the sign of the six red matrix components, leaving everything else the same. Applying this map to any of the three rotation matrices R_{yz}, R_{zx}, R_{xy} has no effect, for example

$$R_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \tilde{R}_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

however, applying the same map to any of the three boost matrices flips two signs, for example

$$B_{tz} = \begin{pmatrix} \cosh\phi_z & 0 & 0 & \sinh\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi_z & 0 & 0 & \cosh\phi_z \end{pmatrix} \rightarrow \tilde{B}_{tz} = \begin{pmatrix} \cosh\phi_z & 0 & 0 & -\sinh\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi_z & 0 & 0 & \cosh\phi_z \end{pmatrix}.$$

Similarly, the basis generators of rotation, T_i (or J_i), remain unaffected, whereas the basis generators of boost, U_i (or K_i), change the overall sign, for example

$$U_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \tilde{U}_z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

See the lower branch of the diagram. Note that the commutation relations of SO⁺(1,3) are still satisfied when changing $U_i \rightarrow -U_i$, as expected for a valid representation.

The observation that parity inversion leaves the generators of rotation unaffected, $T_i \rightarrow T_i$, but changes the signs of the generators of boost, $U_i \rightarrow -U_i$, is crucial because it permits us to parity invert representations that do not act directly on space-time vectors, that is, representations for which the corresponding P matrix is not obvious [PfS, Ch. 3.7.2]. We will see examples of this shortly.

Are the defining representation of SO⁺(1,3) and its parity-inverted version equivalent? Yes, they are! As we have seen, the two representations are related by $\tilde{\Lambda} = P\Lambda P$ and because $P^{-1} = P$ this is a similarity transformation (just like what we had for the dual representation). We say that the defining representation of SO⁺(1,3) *respects parity* and is *not chiral* (= has no handedness). Later, when we come to Weyl spinors, we will encounter inequivalent left- and right-chiral representations.

To *time reverse* a 4-vector we change the sign of its time coordinate. In other words, given a 4-vector x, the time-reversed 4-vector is Tx, where [QFTGA, Ch. 15.4]

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Interestingly, the map $\tilde{\Lambda} = T\Lambda T$ does exactly the same thing as the map $\tilde{\Lambda} = P\Lambda P = (-T)\Lambda(-T) = T\Lambda T$. So, there is no need to distinguish between time-reversed and parity-inverted representations and from now on, we refer only to the latter.

Incidentally, having defined the parity-inversion operator P and the time-reversal operator T, we can now express the full Lorentz group O(1,3) as {SO⁺(1,3), P SO⁺(1,3), T SO⁺(1,3), PT SO⁺(1,3)}, explicitly listing its four components [QFTGA, Ch. 15.4]. Here, P turns proper rotations into improper (reflected) rotations and T turns orthochronous boosts into an antichronous (time-reversed) boosts by multiplying the elements of SO⁺(1,3) from the left. Furthermore, we can express SO(1,3) as {SO⁺(1,3), PT SO⁺(1,3)}.