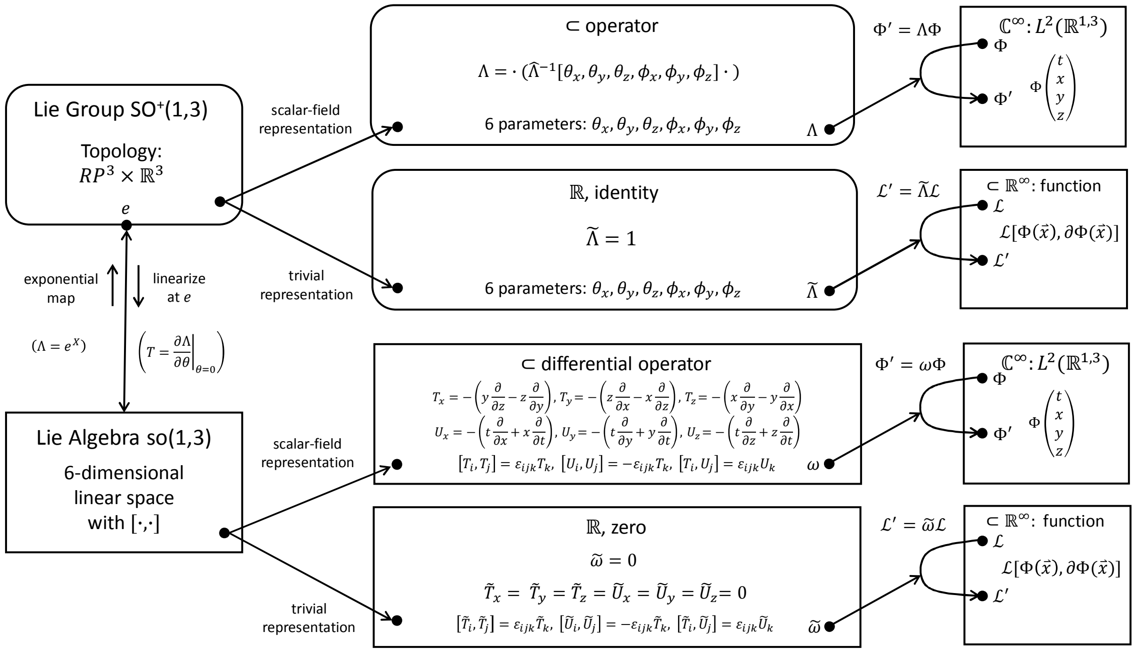
5.15 SO⁺(1,3): Application to Scalar-Field Dynamics; Klein-Gordon Lagrangian



How does a free (= non-interacting) scalar field evolve in time? We know from physics that time evolution is governed by an action principle. To satisfy the requirements of special relativity, the associated action must remain invariant under Lorentz transformations. In other words, whereas the scalar field furnishes an infinite-dimensional representation of SO⁺(1,3), the action functional must furnish the *trivial representation* of SO⁺(1,3). The action can be written as the integral of a Lagrangian density over space time [TM, Vol. 3]. Therefore, if the Lagrangian density is invariant, so is the action. The upper branch of the diagram shows again the scalar-field representation of SO⁺(1,3), where $\hat{\Lambda}$ is the defining representation, and the lower branch shows the trivial representation acting on the Lagrangian density $\mathcal{L}$, which is a real function of the field, $\Phi(\vec{x})$, and its derivatives (gradient), $\partial\Phi(\vec{x})$.

What candidate terms for the Lagrangian density, which must remain invariant under Lorentz transformations, can we construct from the scalar field and its derivatives? In principle there are infinitely many possibilities, but the following two additional constraints narrow them down to just a few. First, we consider only terms with *first-order derivatives* (it turns out that this is necessary for the energy to be bounded from below, that is, to ensure stability), second, we consider only field products up to *second order* (it turns out that this is necessary for the field to be free, that is, not to interact with itself) [PfS, Ch. 4.2].

From the scalar field itself, Φ, we can construct the following invariant terms that satisfy the above constraints: $A + B\Phi(\vec{x}) + C\Phi^2(\vec{x})$, where A, B, C are real constants. Because the equation of motion, which is derived from the Lagrangian density by making the action stationary (usually minimal), does *not* depend on a constant term like A, we can drop it. Furthermore, it turns out that a linear term like $B\Phi(\vec{x})$ introduces only an additive constant to the field value in the equation of motion, which is irrelevant in the case of free fields [PfS, Ch. 6.2]. Thus, the only surviving term is the square term: $C\Phi^2(\vec{x})$.

The gradient of the scalar field is $\partial_\mu \Phi(\vec{x})$, where we switched to tensor-index notation. To construct an invariant from that, we identify this 4-covector with the 4-vector $\partial^\mu \Phi = \eta^{\mu\nu}\partial_\nu \Phi$, where $\eta^{\mu\nu}$ is the Minkowski metric, and then act with the gradient 4-covector on this 4-vector: $\partial_\mu \Phi \partial^\mu \Phi = \eta^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi$ [PfS, Ch. 6.2]. (Incidentally, a dimensional analysis shows that terms of higher order than $\partial_\mu \Phi \partial^\mu \Phi$ must depend on an energy scale. At low energies, the coefficients of these terms become small and can be neglected [NNQFT, Ch. 17.1].)

Thus, the general Lagrangian density for a scalar field transforming under the trivial representation of SO⁺(1,3) and satisfying our additional constraints is $a\partial_\mu \Phi(\vec{x})\partial^\mu \Phi(\vec{x}) + b\Phi^2(\vec{x})$, where a and b are real constants. It is conventional to call $b/a = -m^2$ and choose $a = 1/2$ (an overall scale factor for the Lagrangian density has no effect on the equation of motion and thus can be chosen arbitrarily). After these substitutions, we arrive at the standard form of the (real) Klein-Gordon Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \Phi(\vec{x})\partial^\mu \Phi(\vec{x}) - m^2\Phi^2(\vec{x})\right].$$

For the equation of motion that follows from this Lagrangian density as well as the solutions of that equation see the Appendix "Decoding the Klein-Gordon Equation".

For a complex scalar field, $\Phi \in \mathbb{C}$, the terms must be slightly modified to ensure that the Lagrangian density (and thus the action) remains real. Instead of Φ^2, we use $|\Phi|^2 = \Phi^*\Phi$ and instead of $\partial_\mu \Phi \partial^\mu \Phi$ we use $\partial_\mu \Phi^* \partial^\mu \Phi$. The standard form of the *complex* Klein-Gordon Lagrangian density is

$$\mathcal{L} = \partial_\mu \Phi^*(\vec{x})\partial^\mu \Phi(\vec{x}) - m^2\Phi^*(\vec{x})\Phi(\vec{x}).$$

If we plug the field $\Phi := (\Phi_r + i\Phi_i)/\sqrt{2}$ into the complex Klein-Gordon Lagrangian density, it breaks up into the sum of two real Klein-Gordon Lagrangian densities (with the same mass parameter m), one for the real part, Φ_r, and one for the imaginary part, Φ_i, of the field [PfS, Ch. 6.2.1; QFTGA, Ch. 7.5].

The Higgs field is a scalar field in the sense that it remains invariant under a Lorentz transformation (= a Lorentz scalar). In contrast to the field described by the above Klein-Gordon equation, it is a complex *doublet*, $\Phi \in \mathbb{C}^2$, and *interacts* with itself (i.e., is not free). The Higgs-field Lagrangian density can be written as

$$\mathcal{L} = \partial_\mu \Phi^\dagger(\vec{x})\partial^\mu \Phi(\vec{x}) + \lambda\left(\Phi^\dagger(\vec{x})\Phi(\vec{x}) - \Phi_0^2\right)^2.$$

The potential $V(\Phi) = -\lambda\left(\Phi^\dagger\Phi - \Phi_0^2\right)^2 = -\lambda\Phi^\dagger\Phi\Phi^\dagger\Phi + 2\lambda\Phi_0^2\,\Phi^\dagger\Phi - \lambda\Phi_0^4$ has the shape of a (higher-dimensional) "Mexican hat" and the 4th-order term $\lambda\Phi^\dagger\Phi\Phi^\dagger\Phi$ is responsible for the self-interaction. The vacuum (expectation) value of the Higgs field is nonzero and given by $|\Phi(\vec{x})| = \Phi_0$.