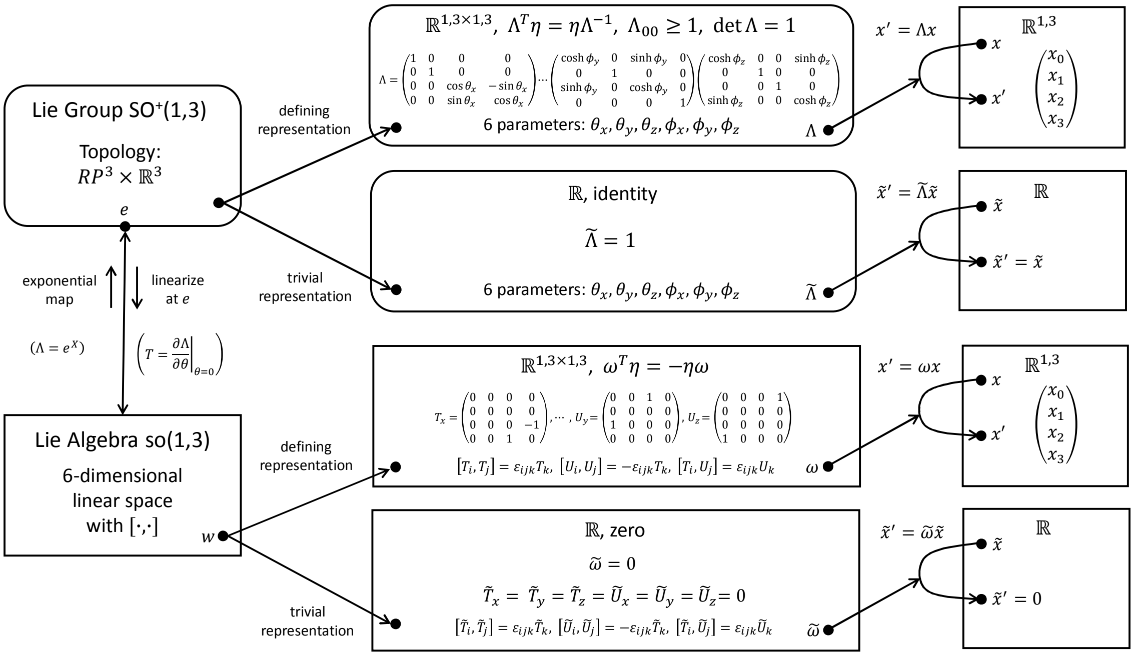
5.11 SO⁺(1,3): The Group of Proper Orthochronous Lorentz Transformations



Moving on to space-time, our representation now acts on the 4-vector $x = (x_0, x_1, x_2, x_3)^T = (t, x, y, z)^T$, which describes a space-time *event* or, when referring to the difference of two events, a space-time *interval*. For clarity, we will sometimes put a harpoon on top of 4-vectors: $\vec{x} = (t, x, y, z)^T$ (this is *not* standard notation). Whereas the invariant distance measure for 4D Euclidean space was given by $d^2 = w^2 + x^2 + y^2 + z^2$, for space-time according to the theory of *special relativity* (= *Minkowski space*) it is given by $\tau^2 = t^2 - x^2 - y^2 - z^2$, where τ is known as the *proper time*. (To keep our equations simple, we use units in which the speed of light is one, $c = 1$.) Defining the *Minkowski metric* $\eta = \mathrm{diag}(+1, -1, -1, -1)$, we can write the (squared) proper time as

$$\tau^2 = x^T\eta x = (t \quad x \quad y \quad z)\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = t^2 - x^2 - y^2 - z^2.$$

In contrast, the Euclidean distance was given by $d^2 = x^T x = x^T I x$, where $I = \mathrm{diag}(1, 1, 1, 1)$ is the metric. See the Appendix "Euclid, Galileo, Einstein: Distance in Space and Time" for a comparison.

What transformations Λ leave the proper time invariant? The condition $x'^T\eta x' = x^T\eta x$ can be expanded to $(\Lambda x)^T\eta(\Lambda x) = x^T\eta x$ or, equivalently, to $x^T\Lambda^T\eta\Lambda x = x^T\eta x$. Hence, the defining condition for the so-called *Lorentz transformations* is $\Lambda^T\eta\Lambda = \eta$ or, equivalently, $\Lambda^T\eta = \eta\Lambda^{-1}$. (Note that for $\eta \to I$ and $\Lambda \to R$, we recover the orthogonal transformations: $R^T = R^{-1}$.) A general Lorentz transformation can be written as the matrix product $\Lambda(\theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z) = R_{yz}(\theta_x) \cdot R_{zx}(\theta_y) \cdot R_{xy}(\theta_z) \cdot B_{tx}(\phi_x) \cdot B_{ty}(\phi_y) \cdot B_{tz}(\phi_z)$, where [QFTGA, Ch. 9.5]

$$R_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}, \; R_{zx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}, \; R_{xy} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & -\sin\theta_z & 0 \\ 0 & \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{tx} = \begin{pmatrix} \cosh\phi_x & \sinh\phi_x & 0 & 0 \\ \sinh\phi_x & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B_{ty} = \begin{pmatrix} \cosh\phi_y & 0 & \sinh\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi_y & 0 & \cosh\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B_{tz} = \begin{pmatrix} \cosh\phi_z & 0 & 0 & \sinh\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi_z & 0 & 0 & \cosh\phi_z \end{pmatrix}.$$

See the upper branch of the diagram. The first three matrices describe the familiar rotations in the yz, zx, and xy planes; the second three matrices are new and represent "hyperbolic rotations" in the three space-time planes tx, ty, and tz. These "hyperbolic rotations" are referred to as *boosts*.

What is the physical meaning of these boosts? The history of a (point-like) object can be described by a parametrized set of events, $\vec{x}(\lambda)$, known as a *world line*. The boost matrix $B_{tx}(\phi_x)$, for example, transforms the events on a word line for an object at rest to those of an object moving along the positive x axis with the constant speed v_x. The parameter ϕ_x, known as the *rapidity*, is related to the speed like $\phi_x = \tanh^{-1}(v_x/c)$. (In this paragraph, we do *not* set $c = 1$ to show its effect.) Unlike an angle, the rapidity ϕ_x ranges from zero (for $v_x = 0$) to infinity (for $v_x = c$) and for low speeds ($v_x \ll c$) can be approximated as $\phi_x \approx v_x/c$. Unlike velocities, rapidities add when composing two boosts (in the same direction), just like angles add when composing two rotations (about the same axis). Rewriting B_{tx} in terms of v_x and taking the limit for low speeds, $(1 - v_x^2/c^2)^{-1/2} \approx 1$ and $v_x/c^2 \ll t/x$, we find

$$B_{tx} = \begin{pmatrix} \cosh\phi_x & (\sinh\phi_x)/c & 0 & 0 \\ (\sinh\phi_x)\cdot c & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{1 - v_x^2/c^2}} \begin{pmatrix} 1 & v_x/c^2 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the last matrix describes the nonrelativistic boost $t' = t$ and $x' = x + v_x t$. See the Appendix "Time Dilation, Lorentz Contraction, and Rest Energy" for effects that occur at high speeds.

Similar to how the group O(4) consisted of proper and improper (reflected) rotations, so does the group O(1,3) consist of proper and improper as well as orthochronous and antichronous (time-reversed) Lorentz transformations. Topologically, the group manifold has *four* components. In contrast, O(4) had only *two* components. Just like the subgroup SO(4) singled out the proper rotations, so does the subgroup SO⁺(1,3) single out the proper orthochronous Lorentz transformations. To restrict to the proper transformations, which do not change the spatial handedness, we require $\det(\Lambda_{ij}) = 1$ with $i, j = 1,2,3$ and to restrict to the orthochronous transformations, which do not reverse the time direction, we require $\Lambda_{00} \geq 1$ [PfS, Ch. 3.7]. Another important topological difference to SO(4) is that SO⁺(1,3) is noncompact. Whereas the space rotations of SO⁺(1,3) are compact, the "hyperbolic space-time rotations" go off to infinity, making the Lorentz group as a whole *noncompact*.

The six basis generators of the Lie algebra so(1,3) can be found by taking the derivatives $T_i = \partial\Lambda/\partial\theta_i$ and $U_i = \partial\Lambda/\partial\phi_i$ at $\theta_i = \phi_i = 0$. Note that the generators of rotation, T_i, are antisymmetric, as usual, but the generators of boost, U_i, are *symmetric* (see the diagram). A general element of the Lie algebra can be written as $\omega = T_i\theta_i + U_i\phi_i$ (summation implied) and satisfies $\omega^T\eta = -\eta\omega$. In contrast to so(4), a general generator of so(1,3) is *not* antisymmetric (but $\eta\omega$ and $\omega\eta$ are). The commutation relations are almost the same as for so(4) except for one crucial difference: the minus sign in $[U_i, U_j] = -\varepsilon_{ijk}T_k$. Like for so(3), the generators of so(1,3) can be associated with observables: T_i with the *angular momentum* and U_i with the *center of mass motion* [RtR, Ch. 18.6-7; PfS, Ch. 4.6-7].

Like any group, the Lorentz group has a trivial representation, which is shown in the lower branch of the diagram. For example, the proper time τ transform under the trivial representation (i.e., it is invariant).