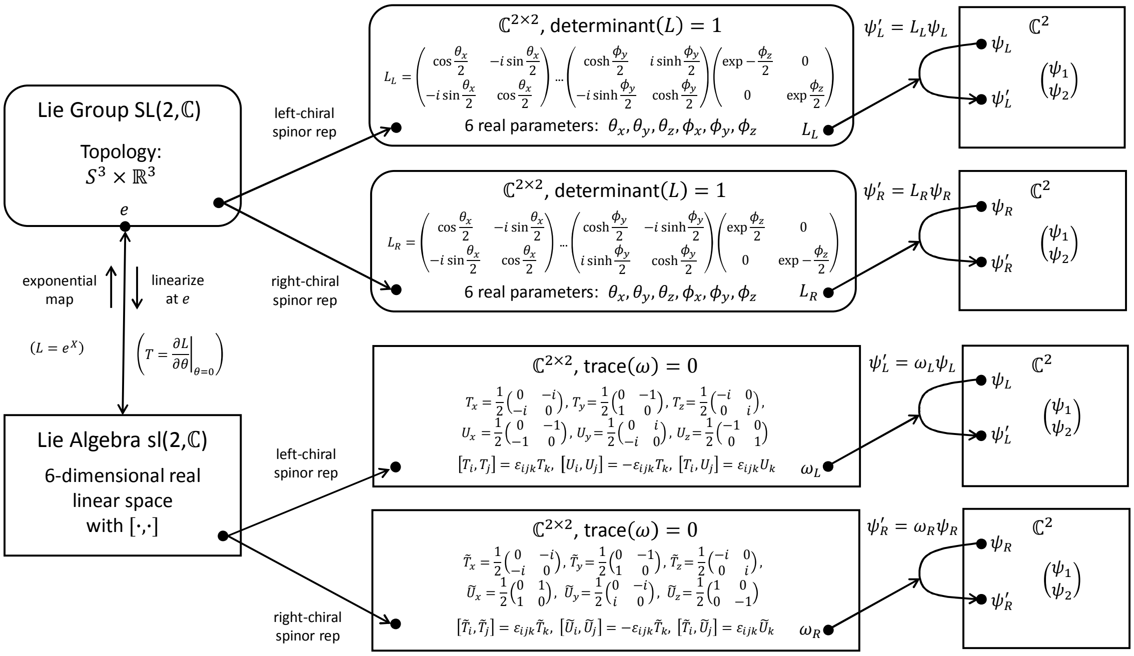
5.23 SL(2,ℂ): Left- and Right-Chiral Weyl-Spinor Representations; (½, 0) and (0, ½)



Earlier, we discussed how to obtain the parity-inverted representation by changing the sign of the three boost generators U_i. Then, we saw that for the defining representation of SO⁺(1,3) parity inversion yields a representation that is *equivalent* (= related by a similarity transformation) to the original one. What happens in the case of SL(2,ℂ)? As we will see in a moment, parity inversion yields a new, *inequivalent* representation! This means that SL(2,ℂ) has *two* 2-dimensional representations. The one we discussed in the previous example is known as the *left-chiral* (or *left-handed*) spinor representation (shown again in the upper branch of the diagram; $T_i = -i\sigma_i/2$, $U_i = -\sigma_i/2$) and the new one, which we'll discuss in the following, is known as the *right-chiral* (or *right-handed*) spinor representation (lower branch of the diagram; $T_i = -i\sigma_i/2$, $U_i = +\sigma_i/2$). These names allude to the handedness of the corresponding 4-vector representations of SO⁺(1,3). The first representation, acting on left-chiral spinors, ψ_L, is labeled (½, 0) and the second one, acting on right-chiral spinors, ψ_R, is labeled (0, ½). Collectively, the two types of spinors are known as *Weyl spinors*.

How are the two representations related? A general generator of the left-chiral spinor representation can be written as $\omega_L = \begin{pmatrix} z & x \\ y & -z \end{pmatrix}$, where x, y, and z are (arbitrary) complex numbers. Flipping the sign of the basis generators U_i, we find the corresponding generator of the right-chiral spinor representation $\omega_R = -\begin{pmatrix} z^* & y^* \\ x^* & -z^* \end{pmatrix}$, where the star indicates complex conjugation. These two generators are related by $\omega_R = -\omega_L^\dagger$. It turns out that this can also be written as $\omega_R = \varepsilon^{-1}\omega_L^*\varepsilon$:

$$\omega_R = \varepsilon^{-1}\omega_L^*\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^* & x^* \\ y^* & -z^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} z^* & y^* \\ x^* & -z^* \end{pmatrix},$$

where ε is the 2-dimensional (rank-2) Levi-Civita symbol ε_{ij}. This is *not* a similarity transformation and hence the two representations are *inequivalent*. But the right-chiral representation is equivalent to the *complex-conjugate representation* of the left-chiral representation. Exponentiating the generators, we find that the corresponding group transformations are related by $L_R = L_L^{-1\dagger}$ or, equivalently, by $L_R = \varepsilon^{-1} L_L^* \varepsilon$. The reverse relationship is given by $\omega_L = -\omega_R^\dagger$ or $\omega_L = \varepsilon^{-1}\omega_R^*\varepsilon$ and $L_L = L_R^{-1\dagger}$ or $L_L = \varepsilon^{-1} L_R^* \varepsilon$, where we used that $\varepsilon^{-1} = -\varepsilon$.

How are the left- and right-chiral spinors related? We know that the generator ω_L acts on a left-chiral spinor like $\psi_L' = \omega_L \psi_L$ and that we can relate it to its right-chiral cousin by $\omega_L = \varepsilon^{-1}\omega_R^*\varepsilon$. Plugging the second equation into the first one, we find that $\psi_L' = \varepsilon^{-1}\omega_R^*\varepsilon\psi_L$. Left-multiplying this equation by ε and complex conjugating it reveals $\varepsilon\psi_L'^* = \omega_R \varepsilon \psi_L^*$, which means that $\varepsilon\psi_L^*$ transforms just like ψ_R! Similarly, we can show that $\varepsilon\psi_R^*$ transforms just like ψ_L. In summary, the two types of spinors are related by complex conjugation and multiplication by ε. (The *van der Waerden notation* keeps track of these operations with dotted and undotted as well as upstairs and downstairs indices [PfS, Ch 3.7.7].)

From the two types of spinors, we can construct several invariants, which will become important later. For example, we can take the Hermitian product of a left- and right-chiral spinor or the other way around: $\chi_L^\dagger \psi_R$ or $\chi_R^\dagger \psi_L$. To demonstrate that $\chi_L'^\dagger \psi_R' = \chi_L^\dagger \psi_R$, we expand the left-hand side to $(L_L \chi_L)^\dagger L_R \psi_R$ and transpose it to $\chi_L^\dagger L_L^\dagger L_R \psi_R$, which equals $\chi_L^\dagger \psi_R$ (= the right-hand side) because $L_L^\dagger L_R = I$. Moreover, using the facts that $\varepsilon\psi_L^*$ transforms like ψ_R and $\varepsilon\psi_R^*$ transforms like ψ_L, we can construct two more invariants from spinor pairs of the same chirality, namely $\chi_L^T \varepsilon \psi_L$ and $\chi_R^T \varepsilon \psi_R$. Because the matrix ε in these spinor expressions plays a similar role as the metric tensor η in 4-vector expressions, it is known as the *spinor metric*. Note that if we restrict SL(2,ℂ) to the subgroup SU(2) by setting $\phi_x = \phi_y = \phi_z = 0$, the distinction between the two 2-dimensional representations disappears, $L_L = L_R$, $\omega_L = \omega_R$, and $\psi_L = \psi_R = \psi$, and the above four invariants reduce to $\chi^\dagger \psi$ and $\chi^T \varepsilon \psi$, which we already know from SU(2).

Let's take a spin-½ particle in the horizontal state $1/\sqrt{2}\,(1,1)^T$ and boost it in the positive z direction close to the speed of light. If the particle is left chiral, we use the transformation $L_L(\phi_z \to \infty)$ and get (after normalization) the state $(0,1)^T$, which represents "spin down" along the z axis. The spin has become *antiparallel* to the direction of boost, in agreement with our earlier example. Next, if the particle is right chiral, we use the transformation $L_R(\phi_z \to \infty)$ and get (after normalization) the state $(1,0)^T$, which represents "spin up". Now, the spin becomes *parallel* to the direction of boost.

Helicity is defined as the spin component in the direction of motion (= the projection of the spin vector on a unit vector in the direction of motion). *Chirality* is defined by how the particle's state transforms under a boost: according to L_L or L_R. In the above example the particle with left chirality acquired a helicity of $-1/2$ and the particle with right chirality acquired a helicity of $+1/2$.

According to the Dirac equation only massless particles have a definite and time-independent chirality, as we will see later. Moreover, massless particles always move at the speed of light. Thus, massless left-chiral particles always have negative helicity and massless right-chiral particles always have positive helicity. Surprisingly, neutrinos (which are approximately massless) come *only* in a the left-chiral form, no right-chiral neutrino has ever been observed! Even more surprisingly, left-chiral fermions (neutrinos, electrons, etc.) carry a charge (weak isospin charge) whereas right-chiral ones do not!!