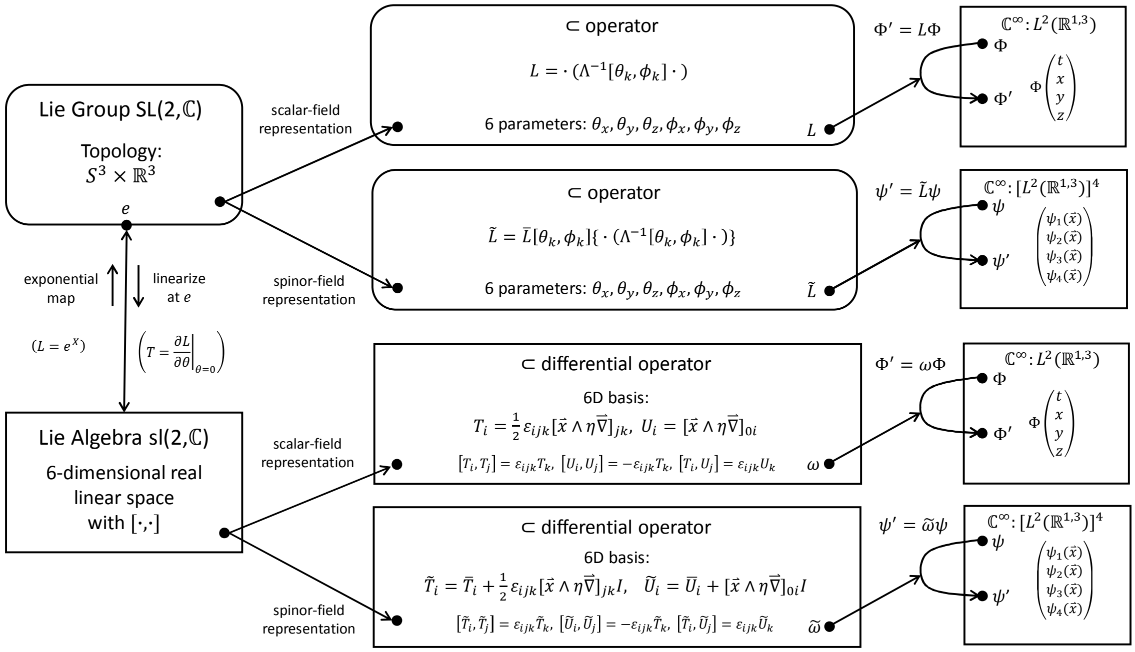
5.27 SL(2,ℂ): Spinor-Field Representation



After discussing scalar and vector fields, we now turn to *spinor fields*. We consider the spinor field $\psi(\vec{x})$ where $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is a Dirac spinor (either in the chiral or mass basis) that is a function of space-time $\vec{x} = (t, x, y, z)^T$. An example of such a spinor field is the electron/positron field in quantum-field theory. The upper branch of the diagram shows a (complex) scalar-field representation of SL(2,ℂ) for reference and the lower branch shows the spinor-field representation, which we will discuss now.

How does a Dirac-spinor field transform under SL(2,ℂ)? Having discussed the 4-vector field earlier, we know how to proceed. A Dirac spinor by itself transforms like $\psi' = \bar{L}\psi$, where $\bar{L}$ is the Dirac-spinor representation of SL(2,ℂ), either in the chiral or mass basis, that we discussed in the previous examples. A scalar field transforms like $\Phi'(\vec{x}) = \Phi(\Lambda^{-1}\vec{x})$, where Λ is the defining representation of SO⁺(1,3), which is also a representation of SL(2,ℂ). Combining the two parts, we conclude that a Dirac-spinor field transforms like $\psi'(\vec{x}) = \bar{L}\psi(\Lambda^{-1}\vec{x})$. Using our informal dot notation to separate the operator from the Dirac-spinor field, $\psi(\vec{x})$, we can write $\tilde{L} = \bar{L}\,\{\cdot\ (\Lambda^{-1}\ \cdot)\}$, where the first dot is a placeholder for the field's name and the second dot is a placeholder for its space-time argument. This operator acts on the Dirac-spinor field like $\tilde{L}\psi(\vec{x}) = \bar{L}\psi(\Lambda^{-1}\vec{x})$ (see the lower branch of the diagram).

What is the corresponding Lie-algebra representation? From our earlier discussion of the 4-vector field, we know that the generators will have two parts: a part for the field's internal structure and a part for the field's space-time structure. The space-time structure is the same as before, but the field's internal structure is now a Dirac spinor instead of a four vector. The six basis generators thus are $\tilde{T}_i = \bar{T}_i + \frac{1}{2}\varepsilon_{ijk}[\vec{x} \wedge \eta\vec{\nabla}]_{jk}I$ (summation over j and k implied) and $\tilde{U}_i = \bar{U}_i + [\vec{x} \wedge \eta\vec{\nabla}]_{0i}I$, where $\bar{T}_i$ and $\bar{U}_i$ are the 4×4 generators acting on Dirac spinors that we discussed earlier (either in the chiral or mass basis) and I is the 4×4 identity matrix, which multiplies the differential operators of the scalar-field representation to make them compatible with the matrices of the Dirac-spinor representation (see the lower branch of the diagram).

For example, the generator $\tilde{T}_x$ for a rotation about the x axis evaluates to

$$\tilde{T}_x = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} + \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} & -\frac{i}{2} & 0 & 0 \\ -\frac{i}{2} & z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} & -\frac{i}{2} \\ 0 & 0 & -\frac{i}{2} & z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \end{pmatrix}.$$

Similarly, the generator $\widetilde{U}_z$ for a boost in the z direction (in the chiral basis) evaluates to

$$\widetilde{U}_z = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \left(-z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z}\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} & 0 & 0 & 0 \\ 0 & \frac{1}{2} - z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & \frac{1}{2} - z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} - z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} \end{pmatrix}.$$

Similar to what we did for the scalar and vector fields, we can rewrite the six basis generators of the Dirac-spinor field representation as $\tilde{T}_i = \bar{T}_i + \frac{1}{2}\varepsilon_{ijk}W^{jk}I$ and $\widetilde{U}_i = \overline{U}_i + W^{0i}I$, where $W^{\mu\nu} = x^\mu\partial^\nu - x^\nu\partial^\mu$. Here we used tensor-index notation for the space-time part.

How can we express a general generator $\widetilde{\omega}$ of the Dirac-spinor-field representation? Given the generator $\widehat{\omega}_{\mu\nu}$ of the 4-vector representation, we know that we can express the corresponding generator of the Dirac-spinor representation as $\frac{1}{4}\widehat{\omega}_{\mu\nu}\gamma^\mu\gamma^\nu$ and that of the scalar-field representation as $\frac{1}{2}\widehat{\omega}_{\mu\nu}W^{\mu\nu}$. Thus, the general generator of the Dirac-spinor-field representation is $\widetilde{\omega} = \frac{1}{4}\widehat{\omega}_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{1}{2}\widehat{\omega}_{\mu\nu}W^{\mu\nu}I$. Putting this generator into the exponential map, yields the associated Dirac-spinor-field transformation $\tilde{L} = \exp\left(\widehat{\omega}_{\mu\nu}\left[\frac{1}{4}\gamma^\mu\gamma^\nu + \frac{1}{2}W^{\mu\nu}I\right]\right)$.