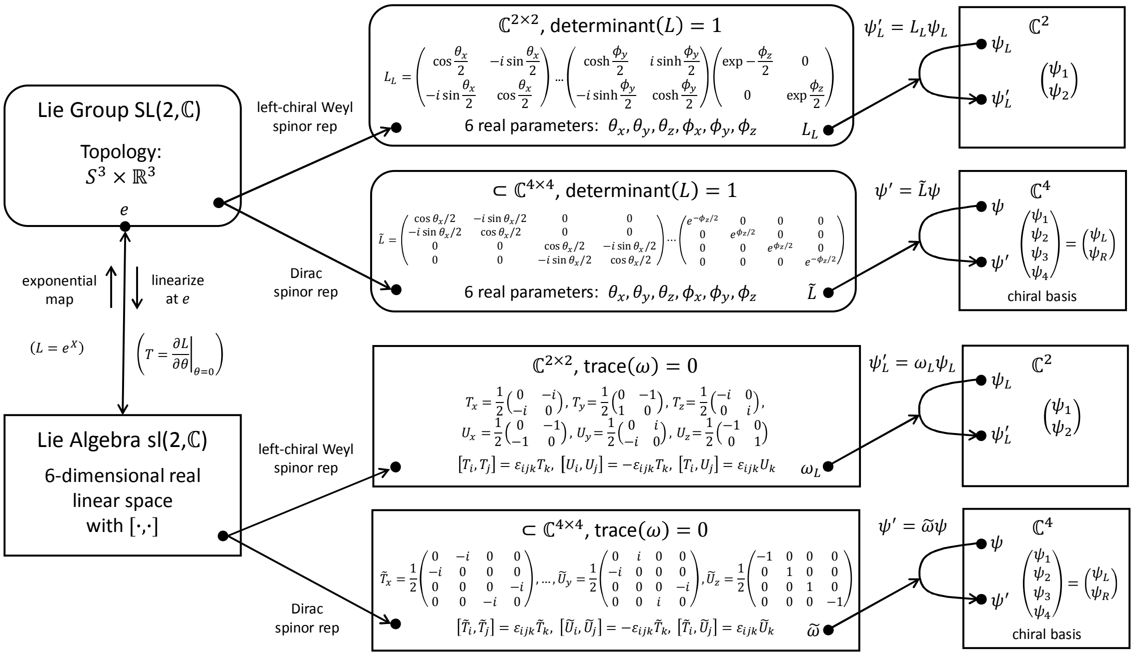
5.24 SL(2,ℂ): Dirac-Spinor Representation; (½, 0) ⊕ (0, ½)



Given the left- and right-chiral Weyl-spinor representations, (½, 0) and (0, ½), we can easily construct the direct sum, (½, 0) ⊕ (0, ½), which is known as the Dirac-spinor representation. It acts on 4-component spinors known as *Dirac spinors* (not to be confused with 4-vectors!). This representation is important because it describes massive fermions like electrons and quarks. The upper branch of the diagram shows again the left-chiral Weyl-spinor representation for reference and the lower branch shows the new Dirac-spinor representation (in the chiral basis).

The simplest way to construct a Dirac spinor is to stack a left-chiral spinor on top of a right-chiral spinor, $\psi = (\psi_L, \psi_R)^T$. Dirac spinors of this form are said to be in the *chiral basis*, a.k.a. *Weyl basis.* (In the next example we will consider another possible basis.) The transformation matrices acting on this representation are obtained by combining a left- and right-chiral spinor transformation matrix in a block-diagonal manner. Note that this block-diagonal format is the hallmark of a reducible representation. The generator matrices are obtained in the same way. The six basis generators, written in terms of the three Pauli matrices σ_x, σ_y, and σ_z, are

$$\tilde{T}_x = -\frac{i}{2}\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix},\ \tilde{T}_y = -\frac{i}{2}\begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix},\ \tilde{T}_z = -\frac{i}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix},$$

$$\tilde{U}_x = -\frac{1}{2}\begin{pmatrix}\sigma_x & 0\\ 0 & -\sigma_x\end{pmatrix},\ \tilde{U}_y = -\frac{1}{2}\begin{pmatrix}\sigma_y & 0\\ 0 & -\sigma_y\end{pmatrix},\ \tilde{U}_z = -\frac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix}.$$

The diagram shows (some of) them as explicit 4×4 matrices.

A general generator can be written as the linear combination $\tilde{\omega} = \tilde{T}_i\theta_i + \tilde{U}_i\phi_i$. Given a generator $\hat{\omega}$ of the 4-vector representation, what is the corresponding generator $\tilde{\omega}$ of the Dirac-spinor representation? This is where Dirac's *gamma matrices,* γ^μ, come in! Using tensor-index notation, we can write $\tilde{\omega} = \frac{1}{4}\hat{\omega}_{\mu\nu}\gamma^\mu\gamma^\nu$. The gamma matrices in the chiral basis are defined as [QFTGA, Ch. 36.2]

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \gamma^5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

where I is the 2×2 identity matrix (and the purpose of γ^5 will become clear later). Similar to what we did for the infinite-dimensional representation of SO⁺(1,3), we can define a matrix (of matrices) $W^{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu$ from which we can pick the basis generators like $\tilde{T}_x = W^{23}$, $\tilde{T}_y = W^{31}$, $\tilde{T}_z = W^{12}$, $\widetilde{U}_x = W^{01}$, $\widetilde{U}_y = W^{02}$, $\widetilde{U}_z = W^{03}$ or, written more compactly, $\tilde{T}_i = \frac{1}{2}\varepsilon_{ijk}W^{jk}$ and $\widetilde{U}_i = W^{0i}$.

How does a parity transformation affect the generator $\widetilde{\omega}$? Inspecting the above $\tilde{T}_i$ and $\widetilde{U}_i$ matrices, we see that changing the sign of the three boost generators $\widetilde{U}_i$ exchanges the upper-left and lower-right block matrices. This operation *is* a similarity transformation, as can be demonstrated by

$$\widetilde{\omega}' = \gamma_0^{-1}\widetilde{\omega}\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix},$$

where A and B are general 2×2 sub-matrices (and we switched to "high-school notation" for γ_0). Unlike the Weyl-spinor representations, the Dirac-spinor representation does respect parity!

In the previous example, we constructed four invariants from Weyl spinors: $\chi_L^\dagger\psi_R$, $\chi_R^\dagger\psi_L$, $\chi_L^T\varepsilon\psi_L$, and $\chi_R^T\varepsilon\psi_R$. Now, we are going to construct invariants based on the Dirac spinors χ and ψ. A first invariant is $\chi^\dagger\gamma_0\psi$, which expand to $\chi_L^\dagger\psi_R + \chi_R^\dagger\psi_L$ in terms of Weyl spinors. Defining the *conjugate Dirac spinor* $\bar{\psi} \coloneqq \psi^\dagger\gamma_0$, we can write this invariant more compactly as $\bar{\chi}\psi$. Another possible invariant is $\chi^T\gamma_2\gamma_0\psi$, which expands to $\chi_L^T\sigma_y\psi_L - \chi_R^T\sigma_y\psi_R = -i(\chi_L^T\varepsilon\psi_L - \chi_R^T\varepsilon\psi_R)$ in terms of Weyl spinors. Defining the matrix $C \coloneqq \gamma_2\gamma_0$, we can write this invariant more compactly as $\chi^T C\psi$ [GTNut, Ch. VII.5]. These invariants will become important when we come to the dynamics of the Dirac-spinor field.

Dirac spinors describe massive spin-½ particles, such as electrons and quarks, which necessarily move slower than the speed of light. A physical particle state (= a solution of the Dirac equation) always has a left- *and* a right-chiral part. For states of *definite energy* (= energy or mass eigenstates) the superposition of the two parts is time independent (except for an unobservable phase factor). Besides particles, Dirac spinors also describe *antiparticles*, such as positrons. Specifically, in the rest frame, the (unnormalized) Dirac spinors for (i) an electron with spin up, (ii) an electron with spin down, (iii) a positron with spin up, and (iv) a positron with spin down, all of them with definite energy, are

$$\psi_{e^-\uparrow} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-imt}, \qquad \psi_{e^-\downarrow} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{-imt}, \qquad \psi_{e^+\uparrow} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} e^{imt}, \qquad \psi_{e^+\downarrow} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{imt},$$

where m is the mass and we used units for which $\hbar = c = 1$ [QFTGA, Ch. 36.3; NNQFT, Ch. 5.2.2]. Note that for particles, the two Weyl spinor are equal, $\psi_L = \psi_R$, and for antiparticles they have opposite signs, $\psi_L = -\psi_R$.

The states of definite energy, shown above, are eigenstates of γ_0 with eigenvalue $+1$ for particles and -1 for antiparticles. In contrast, states of *definite chirality* are eigenstates of γ_5 with eigenvalue $+1$ for right chirality and -1 for left chirality. If a particle is forced into a chiral eigenstate, say, into the left-chiral spin-up state $\psi = (1, 0, 0, 0)^T$ at time $t = 0$, its state starts to oscillate at a frequency equal to the particle's mass: $\psi(t) = (1, 0, 0, 0)^T \cos(mt) - i(0, 0, 1, 0)^T \sin(mt)$ [NNQFT, Ch. 5.2.1].