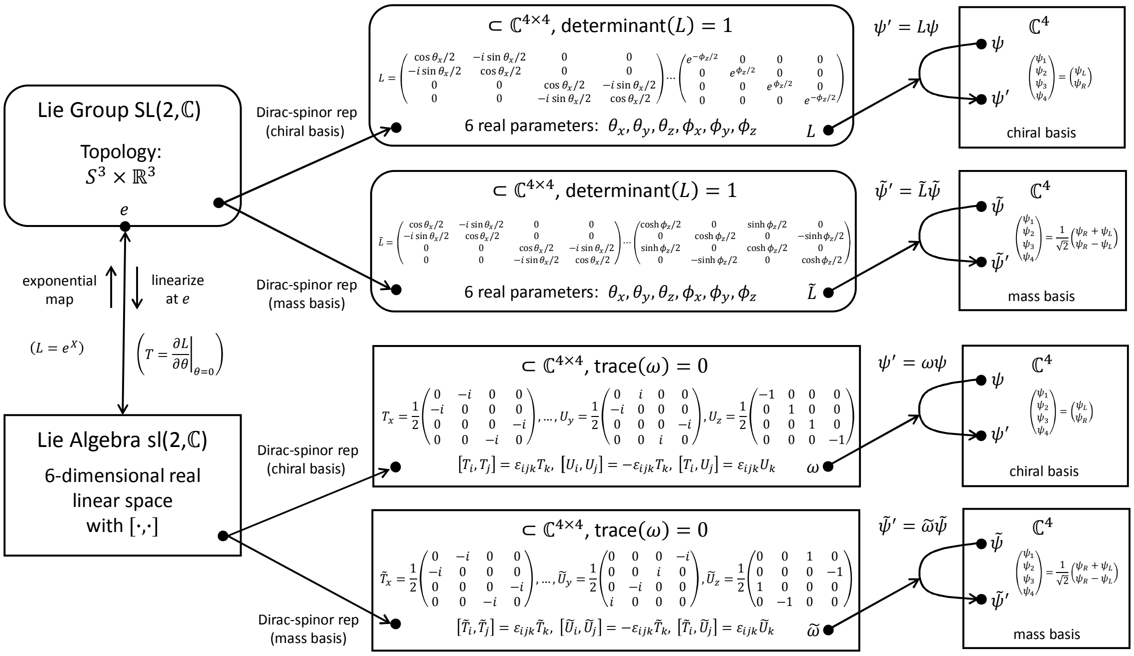
5.25 SL(2,ℂ): Dirac-Spinor Representations in the Chiral and Mass Basis



In the previous example, we saw that particles at rest and in a state of definite energy have identical ψ_L and ψ_R parts. So, rather than constructing the Dirac spinor out of the ψ_L and ψ_R parts directly, we may consider using their sum and difference: $\tilde{\psi} = 1/\sqrt{2}\,(\psi_R + \psi_L, \psi_R - \psi_L)^T$. This amounts to a change of basis, namely from the chiral basis to the so-called *mass basis*, a.k.a. *Dirac basis*. Electroman strolls by and says: "I get it, the upper two components are the chiral common mode and the lower two are the chiral differential mode". The upper branch of the diagram shows again the Dirac-spinor representation in the chiral basis and the lower branch shows the Dirac-spinor representation in the mass basis.

Given a spinor in the chiral basis, we can find the corresponding spinor in the mass basis as $\psi_{mass} = N\psi_{chiral}$ or, the other way around, as $\psi_{chiral} = N^{-1}\psi_{mass}$, where the explicit matrices N and N^{-1} are given by [Wikipedia, "Gamma Matrices"]

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\psi_R + \psi_L\\ \psi_R - \psi_L\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}I & I\\ -I & I\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}, \qquad \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}I & -I\\ I & I\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\psi_R + \psi_L\\ \psi_R - \psi_L\end{pmatrix}.$$

How do the group elements, L, and generators, ω, change when we switch from the chiral to the mass basis? In the chiral basis, the group elements act like $\psi'_{chiral} = L_{chiral}\psi_{chiral}$. Switching to the mass basis, we have $N^{-1}\psi'_{mass} = L_{chiral}N^{-1}\psi_{mass}$ or, equivalently, $\psi'_{mass} = NL_{chiral}N^{-1}\psi_{mass}$ and hence $L_{mass} = NL_{chiral}N^{-1}$. Similarly, the generators change according to $\omega_{mass} = N\omega_{chiral}N^{-1}$. Mapping the basis generators from the chiral basis to the mass basis, we get

$$\tilde{T}_x = -\frac{i}{2}\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}, \tilde{T}_y = -\frac{i}{2}\begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix}, \tilde{T}_z = -\frac{i}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix},$$

$$\tilde{U}_x = \frac{1}{2}\begin{pmatrix}0 & \sigma_x\\ \sigma_x & 0\end{pmatrix}, \tilde{U}_y = \frac{1}{2}\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix}, \tilde{U}_z = \frac{1}{2}\begin{pmatrix}0 & \sigma_z\\ \sigma_z & 0\end{pmatrix}.$$

The diagram shows (some of) them as explicit 4×4 matrices. Note that only the generators of boost, $\widetilde{U}_i$, are affected; the generators of rotation are the same in both bases, $\tilde{T}_i = T_i$.

We know that the basis generators can be expressed as products of two gamma matrices, for example, $\tilde{T}_x = \frac{1}{2}\gamma^2\gamma^3$ or $\widetilde{U}_x = \frac{1}{2}\gamma^0\gamma^1$. Thus, the gamma matrices also change like $\gamma_{mass} = N\gamma_{chiral}N^{-1}$ when switching from the chiral to the mass basis. The explicit gamma matrices in the mass basis are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \gamma^5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Note that only γ^0 and γ^5 are affected; the other gamma matrices remain the same. Independent of the basis, Dirac's gamma matrices always satisfy $(\gamma^0)^2 = +I$, $(\gamma^1)^2 = -I$, $(\gamma^2)^2 = -I$, $(\gamma^3)^2 = -I$, and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$, that is, they generate a Clifford algebra with Lorentzian signature.

Many results that we derived for the chiral basis remain valid in the mass basis. In particular, the similarity transformation $\omega' = \gamma_0^{-1}\omega\gamma_0$ still maps $T_i \to T_i$ and $U_i \to -U_i$ and thus reverses the parity. Instead of swapping the diagonal block matrices of ω, the parity transformation now flips the signs of the off-diagonal block matrices. The expressions $\chi^\dagger\gamma_0\psi = \bar{\chi}\psi$ and $\chi^T\gamma_2\gamma_0\psi = \chi^T C\psi$ are still invariants.

In the mass basis, the Dirac spinors for physical electrons/positrons with spin up/down, which we discussed earlier for the chiral basis, change to

$$\psi_{e^-\uparrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \qquad \psi_{e^-\downarrow} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \qquad \psi_{e^+\uparrow} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt}, \qquad \psi_{e^+\downarrow} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imt},$$

where the electrons/positrons are again taken to be at rest and in an energy eigenstate [NNQFT, Ch. 5.2.3]. Note that these Dirac spinors have only one nonzero entry:

- In the *mass basis*, states of definite energy (= mass) are simple. They are eigenstates of γ_0, which is diagonal in the mass basis. In contrast, states of definite chirality are more complicated.
- In the *chiral basis*, states of definite chirality are simple. They are eigenstates of γ_5, which is diagonal in the chiral basis. In contrast, states of definite energy are more complicated.

Let's boost the Dirac spinor of an electron with spin up and definite energy in the z direction. In the *chiral basis* the spinor at rest is $(1, 0, 1, 0)^T$, where we suppressed the phase factor e^{-imt}. Boosting it in the positive z direction (see the upper branch of the diagram for the relevant boost matrix) yields $(\exp(-\phi_z/2), 0, \exp(\phi_z/2), 0)^T$, where ϕ_z is the rapidity. At low speeds, this can be approximated by $(1 - v_z/(2c), 0, 1 + v_z/(2c), 0)^T$, where v_z is the velocity, and in the ultra-relativistic limit ($v_z \to c$) it becomes $(0, 0, 1, 0)^T$, which is a right-chiral spin-up state [QFTGA, Ch. 36.3]. This agrees with our discussion of Weyl spinors: when approaching the speed of light, positive helicity (spin up and moving upwards) implies right chirality.

Now, switching to the *mass basis*, the same Dirac spinor at rest is $(1, 0, 0, 0)^T$. Boosting it in the positive z direction (see the lower branch of the diagram for the relevant boost matrix) yields $(\cosh(\phi_z/2), 0, \sinh(\phi_z/2), 0)^T$. At low speeds, this can be approximated by $(1, 0, v_z/(2c), 0)^T$ and in the ultra-relativistic limit it becomes $(1, 0, 1, 0)^T$, which is again a right-chiral spin-up state [NNQFT, Ch. 5.2.3].