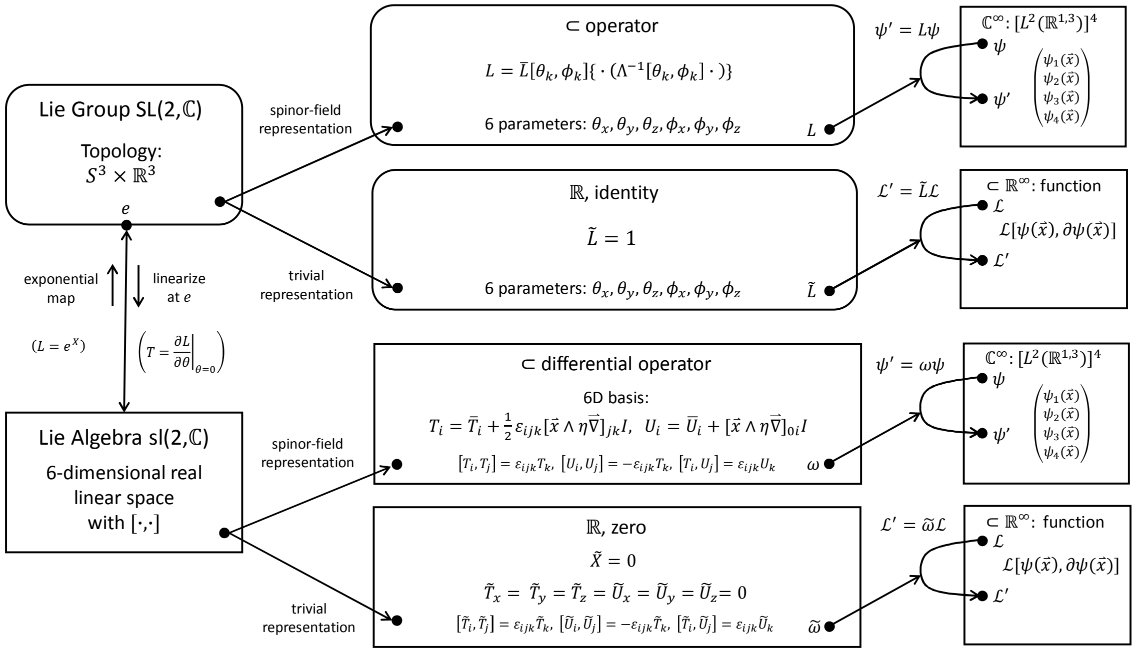
5.28 SL(2,ℂ): Application to Spinor-Field Dynamics; Dirac Lagrangian



How does a free Dirac-spinor field evolve in time? As we discussed for the scalar and vector fields, to satisfy the requirements of special relativity, the associated action must remain invariant under Lorentz transformation. In other words, whereas the Dirac-spinor field furnishes an infinite-dimensional representation of SL(2,ℂ), the action functional must furnish the *trivial representation* of SL(2,ℂ). The upper branch of the diagram shows again the Dirac-spinor-field representation, and the lower branch shows the trivial representation acting on the Lagrangian density, which is a real function of the Dirac-spinor field, $\psi(\vec{x})$, and its derivatives, $\partial\psi(\vec{x})$.

What terms can we construct from the Dirac-spinor field and its derivatives that remain invariant under SL(2,ℂ) and thus are candidate terms for the Lagrangian density? As discussed for the scalar and vector fields, we consider only terms with *first-order derivatives* and only field products up to *second order* [PfS, Ch. 4.2].

We already know two invariants that can be constructed from the Dirac-spinor field itself. First, we have $\bar{\psi}(\vec{x})\psi(\vec{x}) = \psi^\dagger(\vec{x})\gamma_0\psi(\vec{x})$, which expands in terms of Weyl spinors (assuming the chiral basis) to $\psi_L^\dagger(\vec{x})\psi_R(\vec{x}) + \psi_R^\dagger(\vec{x})\psi_L(\vec{x})$. This invariant is real, as appropriate for a term that goes into the Lagrangian, because the second part of the expansion is the complex (or Hermitian) conjugate of the first part: $(\psi_L^\dagger\psi_R)^* = \psi_R^\dagger\psi_L$.

Second, we have the invariant $\psi^T(\vec{x})C\psi(\vec{x}) = \psi^T(\vec{x})\gamma_2\gamma_0\psi(\vec{x})$, which expands in terms of Weyl spinors to $\psi_L^T(\vec{x})\varepsilon\psi_L(\vec{x}) - \psi_R^T(\vec{x})\varepsilon\psi_R(\vec{x})$. Expanding this further in terms of spinor components, reveals that it is zero, $(\psi_1\psi_2 - \psi_2\psi_1) - (\psi_3\psi_4 - \psi_4\psi_3) = 0$, making this expression seemingly useless as a candidate term for the Lagrangian. However, in quantum-field theory (QFT) fields are operator valued and do not necessarily commute, invalidating our conclusion that the above expression is zero. It turns out that after adding the Hermitian conjugate to make the overall expression Hermitian, $\psi^T(\vec{x})C\psi(\vec{x}) + h.c.$, this becomes a viable candidate term for a QFT Lagrangian [GTNut, Ch. VII.3 and VII.5].

Before we can construct invariants from spinor-field derivatives, we need to understand how to take such derivatives. It turns out that the operator for space-time derivatives of a Dirac-spinor field is $\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z$, where γ^μ are the gamma matrices and ∂_μ is the usual 4-vector derivative [PfS, Ch. 6.3]. Using tensor-index notation, this operator can be written more compactly as $\gamma^\mu\partial_\mu$ or, if we want to be slick, using the Feynman slash, as $\partial\!\!\!/ \coloneqq \gamma^\mu\partial_\mu$ (the slash should be diagonal, but I don't know how to do that in MS Word). Spelled out explicitly (in the chiral basis), this derivative operator is

$$\begin{pmatrix} \mathbf{0} & I\frac{\partial}{\partial t} + \sigma_x\frac{\partial}{\partial x} + \sigma_y\frac{\partial}{\partial y} + \sigma_z\frac{\partial}{\partial z} \\ I\frac{\partial}{\partial t} - \sigma_x\frac{\partial}{\partial x} - \sigma_y\frac{\partial}{\partial y} - \sigma_z\frac{\partial}{\partial z} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\partial}{\partial t} + \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} - \frac{\partial}{\partial z} & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 & 0 \\ -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} + \frac{\partial}{\partial z} & 0 & 0 \end{pmatrix}.$$

(Gamma matrices cannot only be used to convert four vectors to Dirac spinors but also the other way round. Given the Dirac spinor ψ, the object $\bar{\psi}\gamma^\mu\psi$ transforms like a four vector. However, γ^μ by itself is *not* a four vector [GTNut, Ch. VII.4].)

Now, we are ready to construct invariants that depend on spinor-field derivatives! Plugging the derivative $\chi(\vec{x}) = \gamma^\mu\partial_\mu\psi(\vec{x})$ into the invariant $\bar{\psi}(\vec{x})\chi(\vec{x})$ results in $\bar{\psi}(\vec{x})\gamma^\mu\partial_\mu\psi(\vec{x})$. Another possible invariant is $[\gamma^\mu\partial_\mu\bar{\psi}(\vec{x})]\psi(\vec{x})$, but it turns out that the action based on the latter invariant leads to the same equations of motion; so, by convention, we keep only the first one [PfS, Ch. 6.3]. Moreover, to make that action real, we need to multiply the above invariants by the imaginary unit i.

Thus, the general Lagrangian density for a Dirac-spinor field transforming under the trivial representation of SL(2,ℂ) and satisfying our additional constraints is $ai\bar{\psi}(\vec{x})\gamma^\mu\partial_\mu\psi(\vec{x}) + b\bar{\psi}(\vec{x})\psi(\vec{x}) + c[\psi^T(\vec{x})C\psi(\vec{x}) + h.c.]$, where a, b, and c are real constants.

It is conventional to call $b = -m$ and choose $a = 1$. For common particles, like electrons and quarks, Nature does not make use of the last term, which is known as the *Majorana mass term*, and thus we set $c = 0$. (The Majorana mass term may play a role for neutrinos.) After these substitutions, we arrive at the standard form of the Dirac Lagrangian density:

$$\mathcal{L} = i\bar{\psi}(\vec{x})\gamma^\mu\partial_\mu\psi(\vec{x}) - m\bar{\psi}(\vec{x})\psi(\vec{x}),$$

which encodes the dynamics of a Dirac-spinor field or, after quantization, massive spin-½ particles.

Setting $m = 0$ yields the Weyl Lagrangian density $\mathcal{L} = i\bar{\psi}(\vec{x})\gamma^\mu\partial_\mu\psi(\vec{x})$, which splits into a sum of two independent parts $\mathcal{L} = i\psi_L^\dagger(\vec{x})\bar{\sigma}^\mu\partial_\mu\psi_L(\vec{x}) + i\psi_R^\dagger(\vec{x})\sigma^\mu\partial_\mu\psi_R(\vec{x})$, where $\sigma^\mu = (I, \sigma_x, \sigma_y, \sigma_z)^T$ and $\bar{\sigma}^\mu = (I, -\sigma_x, -\sigma_y, -\sigma_z)^T$. It encodes the dynamics of two dispersion-free spinor fields or, after quantization, massless spin-½ particles. These particles are either left *or* right chiral and move at the speed of light. In contrast, the massive particles of the Dirac equation have mixed chirality and move slower than the speed of light.

For the equations of motion that follow from these Lagrangian densities as well as their solutions, see the Appendix "Decoding the Dirac Equation".