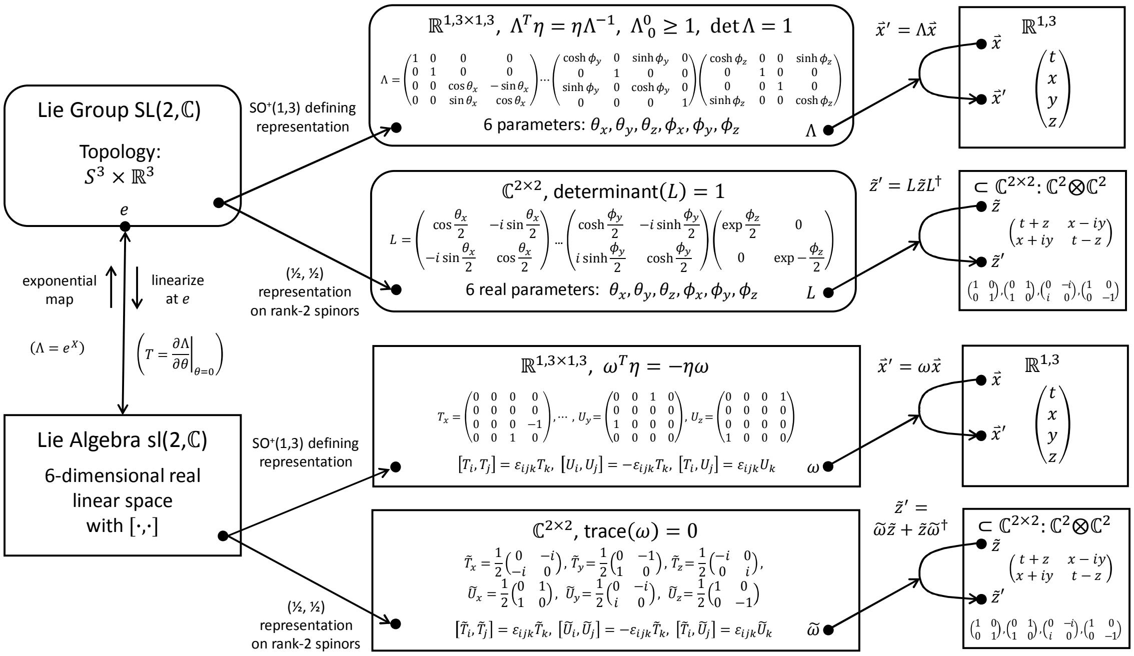
5.26 SL(2,ℂ): Four-Vector Representation; (½, ½) = (½, 0) ⊗ (0, ½)



We'll now turn to the important (½, ½) representation of SL(2,ℂ). To construct it, we take the tensor product (½, 0) ⊗ (0, ½), which acts on rank-2 spinors as shown in the lower branch of the diagram. Given our experience with the (½, ½) representation of Spin(4), we may guess (correctly) that this representation is equivalent to the (complexified) defining representation of SO⁺(1,3). The upper branch of the diagram shows again the defining representation of SO⁺(1,3) for reference.

How does the (½, ½) representation act on the rank-2 spinor $\tilde{z}$? Analogous to our discussion of Spin(4), we use the fact that $\tilde{z}$ transforms like the outer product $\chi_L\psi_R^T$, with $\chi_L' = L_L\chi_L$ and $\psi_R' = L_R\psi_R$, thus $\tilde{z}' = L_L\tilde{z}L_R^T$. Using the relationship $L_R = \varepsilon^{-1}L_L^*\varepsilon$ from three examples ago, this transformation becomes $\tilde{z}' = L_L\tilde{z}\varepsilon^{-1}L_L^\dagger\varepsilon$. Now, it turns out that to get a direct correspondence between the rank-2 spinor and the 4-vector representations, we need to pick the similar transformation $\tilde{z}' = L\tilde{z}L^\dagger$, where L is the SL(2,ℂ) representation that we formerly called L_R (see the lower branch of the diagram). (If we choose $L = L_L$ instead, we get the parity-inverted 4-vector representation.)

How do the generators of this representation act on $\tilde{z}$? Taking the derivative of the transformation $\tilde{z}' = L\tilde{z}L^\dagger$ with respect to the parameters of L (using the product rule) and evaluating the result at the identity, we find $\tilde{z}' = \tilde{\omega}\tilde{z} + \tilde{z}\tilde{\omega}^\dagger$, where $\tilde{\omega}$ is the generator of L (see the lower branch of the diagram; $\tilde{T}_i = -i\sigma_i/2$ and $\tilde{U}_i = \sigma_i/2$ like in the right-chiral Weyl representation).

Analogous to our discussion of Spin(4), we can view the representation space $\mathbb{C}^{2\times2}$ as a 4-dimensional vector space over the complex numbers or as an 8-dimensional vector space over the reals. In the latter case, we have an 8-dimensional representation that is reducible into two 4-dimensional ones. Why? A general complex 2×2 matrix can be decomposed into a Hermitian and an anti-Hermitian part like

$$\tilde{z} + \bar{z} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} + i\begin{pmatrix} \bar{t}+\bar{z} & \bar{x}-i\bar{y} \\ \bar{x}+i\bar{y} & \bar{t}-\bar{z} \end{pmatrix},$$

where $t, x, y, z, \bar{t}, \bar{x}, \bar{y}$, and $\bar{z}$ are 8 real numbers. The Hermitian matrix $\tilde{z}$ (first part) satisfies $\tilde{z} = \tilde{z}^\dagger$ and lives in the 4-dimensional subspace spanned by the basis I, σ_x, σ_y, σ_z. The transformation $\tilde{z}' = L\tilde{z}L^\dagger$ always maps Hermitian matrices to Hermitian ones [PfS, Ch. 3.7.8]. Similarly, the anti-Hermitian matrix $\bar{z}$ (second part) satisfies $\bar{z} = -\bar{z}^\dagger$ and lives in the 4-dimensional subspace spanned by the basis iI, $i\sigma_x$, $i\sigma_y$, $i\sigma_z$. The transformation $\bar{z}' = L\bar{z}L^\dagger$ always maps anti-Hermitian matrices to anti-Hermitian ones. The lower branch of the diagram shows the 4-dimensional representation on Hermitian matrices.

We suspected that the (½, ½) representation of SL(2,ℂ) acting on the 4-dimensional Hermitian (or anti-Hermitian) subspace is equivalent to the defining representation of SO⁺(1,3). Let's check that! Picking a boost in the positive z direction, the transformation $\tilde{z}' = L\tilde{z}L^\dagger$ is

$$\begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} = \begin{pmatrix} e^{\phi_z/2} & 0 \\ 0 & e^{-\phi_z/2} \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \begin{pmatrix} e^{\phi_z/2} & 0 \\ 0 & e^{-\phi_z/2} \end{pmatrix}.$$

Multiplying out the matrices and solving for $(t', x', y', z')^T$, we find that the vector transforms like

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{\phi_z}+e^{-\phi_z} & 0 & 0 & e^{\phi_z}-e^{-\phi_z} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ e^{\phi_z}-e^{-\phi_z} & 0 & 0 & e^{\phi_z}+e^{-\phi_z} \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\phi_z & 0 & 0 & \sinh\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi_z & 0 & 0 & \cosh\phi_z \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix},$$

which is exactly the boost matrix $B_{tz}(\phi_z)$ of the defining representation of SO⁺(1,3)! Note how the exponentials in the SL(2,ℂ) matrix combine into the hyperbolic sines and cosines of the SO⁺(1,3) matrix. Repeating this exercise for the other boosts and rotations reveals that the two representations are, in fact, equivalent.

Why does this work? The SL(2,ℂ) matrices L and $L^\dagger$ have unit determinants by definition: $\det L = \det L^\dagger = 1$. This means that the transformation $\tilde{z}' = L\tilde{z}L^\dagger$ preserves the determinant: $\det\tilde{z}' = \det\tilde{z}$. But what is the determinant of $\tilde{z}$? It is $t^2 - x^2 - y^2 - z^2$, the Minkowski space-time distance (= proper time). This is exactly the invariance that defines the transformations in SO⁺(1,3)!

Just to be sure, let's also check the action of an algebra element. Picking the generator for a boost in the positive z direction, the map $\tilde{z}' = \widetilde{\omega}\tilde{z} + \tilde{z}\widetilde{\omega}^\dagger$ expands to

$$\begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} + \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solving for $(t', x', y', z')^T$ recovers the basis generator U_z of the defining representation of SO⁺(1,3):

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$

In summary, we can Lorentz transform 4-vectors by doing some kind of magic trick: We "fold" the 4-vector into a rank-2 spinor $\tilde{z}$, act on it with the formula $\tilde{z}' = L\tilde{z}L^\dagger$, where L is an SL(2,ℂ) matrix, and then "pull" the transformed 4-vector back out. Voila!

Given the correspondence between 4-vectors and rank-2 spinors, we may wonder if a 4-vector is equivalent to a left- and right-chiral spinors. No! Rank-2 spinors obtained by taking the outer product $\tilde{z} = \chi_L \psi_R^T$ always have $\det\tilde{z} = 0$ and therefore correspond to *null* 4-vectors, not general 4-vectors.