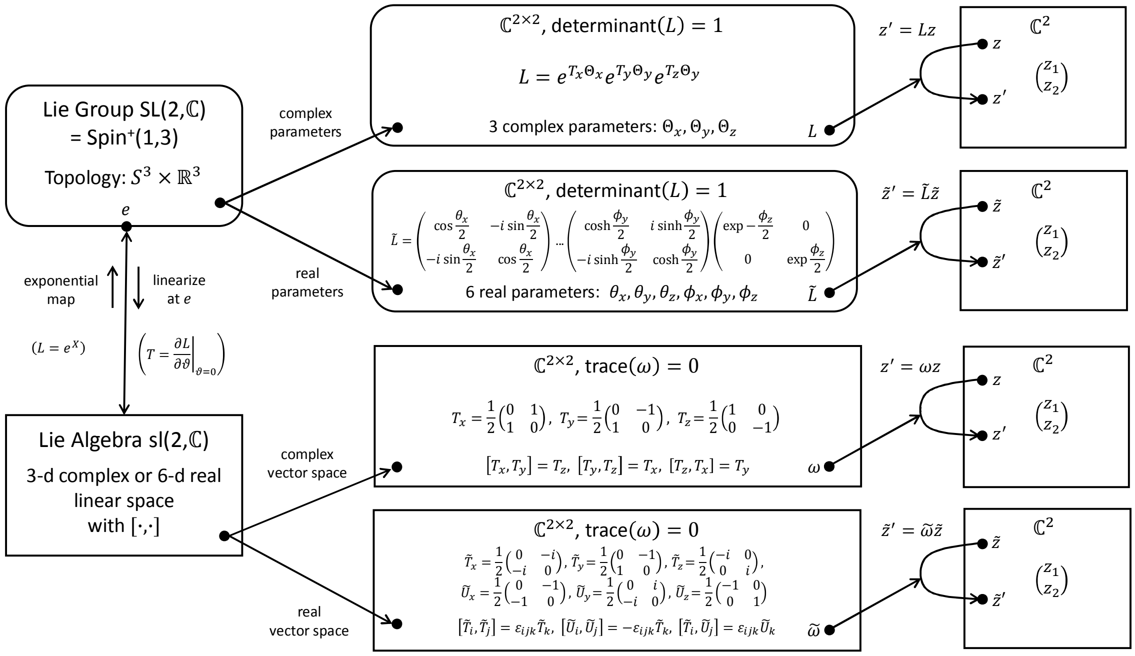
5.22 SL(2,ℂ) = Spin⁺(1,3): The Relativistic Spin Group; Spinor Representations



The Lie group SL(2,ℂ), or rather its defining representation, consists of the complex 2×2 matrices L with determinant one: $L_{11}L_{22} - L_{12}L_{21} = 1$. The S stands for special (= determinant one) and the L stands for linear. It acts on the complex 2-component vectors z, as shown in the diagram. It turns out that SL(2,ℂ) is isomorphic to Spin⁺(1,3), the double cover of SO⁺(1,3)! This is analogous to how SU(2) is isomorphic to Spin(3), the double cover of SO(3). SL(2,ℂ) is known as the *relativistic spin group* [GATP, Lect. 15] and its representations consist of those of the proper orthochronous Lorentz group, SO⁺(1,3), plus the crucial spinor representations, which we will study next.

How many parameters does an SL(2,ℂ) matrix have? A general complex 2×2 matrix has four independent complex parameters, thus an SL(2,ℂ) matrix, which is constrained to have determinant one, has three. Instead of writing the matrix in terms of three complex parameters, we can also write it in terms of six real parameters by splitting each complex parameter into a real and imaginary part. For example, we can write the matrix in terms of the three complex parameters L_{11}, L_{12}, and L_{21} or in terms of the six real parameters L'_{11}, L'_{12}, L'_{21}, L''_{11}, L''_{12}, and L''_{21}:

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & \dfrac{1 + L_{12}L_{21}}{L_{11}} \end{pmatrix} = \begin{pmatrix} L'_{11} + iL''_{11} & L'_{12} + iL''_{12} \\ L'_{21} + iL''_{21} & \dfrac{1 + (L'_{12} + iL''_{12})(L'_{21} + iL''_{21})}{L'_{11} + iL''_{11}} \end{pmatrix}.$$

(This works only if $L_{11} \neq 0$; but if $L_{11} = 0$, we can always find another parametrization that works.) The diagram shows another way to parametrize the same SL(2,ℂ) matrix. In the upper branch it is written in terms of the three complex parameters Θ_x, Θ_y, and Θ_z and in the lower branch in terms of the six real parameters θ_x, θ_y, θ_z, ϕ_x, ϕ_y, and ϕ_z. We'll see how to construct those matrices in a moment.

The Lie algebra sl(2,ℂ) consists of the complex 2×2 matrices ω with trace zero: $\omega_{11} + \omega_{22} = 0$. Like for SU(2), the determinant-one constraint of SL(2,ℂ) translates to the trace-zero constraint of sl(2,ℂ). The space of all ω matrices can be understood as a 3-dimensional vector space over the complex numbers or as a 6-dimensional vector space over the real numbers. In the first case, we have three basis generators, such as the matrices T_x, T_y, and T_z shown in the upper branch of the diagram, which we combine with complex numbers (another possible basis is $T_x \pm T_y$ and T_z). In the second case, we have six basis generators, such as the matrices $\tilde{T}_x$, $\tilde{T}_y$, $\tilde{T}_z$, $\tilde{U}_x$, $\tilde{U}_y$, and $\tilde{U}_z$ shown in the lower branch of the diagram, which we combine with real numbers.

The six basis generators in the lower branch of the diagram were chosen such that the $\tilde{T}_k = -i\sigma_k/2$ are identical to the basis generators we had for su(2) and that the commutation relations of the whole set agree with those of so(1,3) (the sign of $\tilde{U}_k = -\sigma_k/2$ was chosen to agree with the left-spinor definition in [QFTGA Ch. 37.3].) The fact that we can reproduce the commutation relations of so(1,3) shows that the two Lie algebras are isomorphic: sl(2,ℂ) = so(1,3).

The exponential map takes us from sl(2,ℂ) to SL(2,ℂ). We can do this by starting from the 3-dimensional complex parametrization (upper branch) or the 6-dimensional real parametrization (lower branch). In the latter case, a general SL(2,ℂ) transformation with angle parameters θ_k and rapidity parameters ϕ_k can be expressed as the matrix product $\tilde{L}(\theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z) = \tilde{L}_{yz}(\theta_x) \cdot \tilde{L}_{zx}(\theta_y) \cdot \tilde{L}_{xy}(\theta_z) \cdot \tilde{L}_{tx}(\phi_x) \cdot \tilde{L}_{ty}(\phi_y) \cdot \tilde{L}_{tz}(\phi_z) = e^{\tilde{T}_x\theta_x} \cdot e^{\tilde{T}_y\theta_y} \cdot e^{\tilde{T}_z\theta_z} \cdot e^{\tilde{U}_x\phi_x} \cdot e^{\tilde{U}_y\phi_y} \cdot e^{\tilde{U}_z\phi_z}$, where

$$\tilde{L}_{yz} = \begin{pmatrix} \cos\theta_x/2 & -i\sin\theta_x/2 \\ -i\sin\theta_x/2 & \cos\theta_x/2 \end{pmatrix},\ \tilde{L}_{zx} = \begin{pmatrix} \cos\theta_y/2 & -\sin\theta_y/2 \\ \sin\theta_y/2 & \cos\theta_y/2 \end{pmatrix},\ \tilde{L}_{xy} = \begin{pmatrix} \exp -i\theta_z/2 & 0 \\ 0 & \exp i\theta_z/2 \end{pmatrix},$$

$$\tilde{L}_{tx} = \begin{pmatrix} \cosh\phi_x/2 & -\sinh\phi_x/2 \\ -\sinh\phi_x/2 & \cosh\phi_x/2 \end{pmatrix},\ \tilde{L}_{ty} = \begin{pmatrix} \cosh\phi_y/2 & i\sinh\phi_y/2 \\ -i\sinh\phi_y/2 & \cosh\phi_y/2 \end{pmatrix},\ \tilde{L}_{tz} = \begin{pmatrix} \exp -\phi_z/2 & 0 \\ 0 & \exp \phi_z/2 \end{pmatrix}.$$

The matrices $\tilde{L}_{yz}(\theta_x)$, $\tilde{L}_{zx}(\theta_y)$, and $\tilde{L}_{xy}(\theta_z)$ are identical to what we had for SU(2) and describe the usual spinor rotations. The matrices $\tilde{L}_{tx}(\phi_x)$, $\tilde{L}_{ty}(\phi_y)$, and $\tilde{L}_{tz}(\phi_z)$ are new and describe *spinor boosts*. Since Spin⁺(1,3) is the exponential map of so(1,3), the Lie group SL(2,ℂ) is isomorphic to Spin⁺(1,3).

Now we have everything we need to study how the quantum state of a spin-½ particle transforms under a Lorentz boost! As an example, let's boost the state $\tilde{z} = (1,0)^T$ for "spin up" (with respect to the z axis) to half the speed of light (velocity $v = c/2$, rapidity $\phi = 0.55$). If we boost along the x axis using $\tilde{L}_{tx}$, we get $\tilde{z}' = (1.04, -0.28)^T$, which is an (unnormalized) state with a 93.2% probability for "spin up" and a 6.8% probability for "spin down"; boosting along the y axis using $\tilde{L}_{ty}$ yields the state $\tilde{z}' = (1.04, -0.28i)^T$, which has the same probabilities; finally, boosting along the z axis using $\tilde{L}_{tz}$ yields $\tilde{z}' = (0.76, 0)^T$, which is a pure "spin up" state. So, we find that boosting a spin-½ particle orthogonal to its spin axis changes the orientation of the spin axis! In the ultra-relativistic limit (velocity $v \to c$, rapidity $\phi \to \infty$), a boost along the x axis results in the (normalized) spin state $1/\sqrt{2}\,(1,-1)^T$, which corresponds to a spin pointing along the negative x axis, that is, the spin axis has become antiparallel to the direction of boost. The same thing happens for a boost along the y axis.

How does this compare to boosting a 4-vector? If we boost the world line of a stationary vector that points in the z direction, $\vec{x} = (t, 0, 0, 1)^T$, along the x axis it acquires a time-dependent x component. Similarly, if we boost along the y or z axes, it acquires a time-dependent y or z component.