9.6 Noether's Theorem for Particle Theories

Symmetry: Space Translation

$$\left[\frac{d}{d\lambda}L(q_i+\lambda a_i,\dot{q}_i,t)\right]_{\lambda=0}=0$$

$$\frac{\partial L}{\partial q_i}a_i=0$$

Euler-Lagrange: $\frac{\partial L}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)a_i=0$$

Conserved Quantity:

$$p=\frac{\partial L}{\partial \dot{q}_i}a_i$$

Symmetry: Linear Point Transformation

$$\left[\frac{d}{d\lambda}L\big(S_{ij}(\lambda)q_j,S_{ij}(\lambda)\dot{q}_j,t\big)\right]_{\lambda=0}=0$$

$$\frac{\partial L}{\partial q_i}G_{ij}q_j+\frac{\partial L}{\partial \dot{q}_i}G_{ij}\dot{q}_j=0$$

Euler-Lagrange: $\frac{\partial L}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)G_{ij}q_j+\frac{\partial L}{\partial \dot{q}_i}G_{ij}\dot{q}_j=0$$

Product Rule: $\dot{A}B+A\dot{B}=\frac{d}{dt}(AB)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}G_{ij}q_j\right)=0$$

Conserved Quantity:

$$J=\frac{\partial L}{\partial \dot{q}_i}G_{ij}q_j$$

Symmetry: Time Translation

$$\left[\frac{d}{d\lambda}L(q_i,\dot{q}_i,t+\lambda)\right]_{\lambda=0}=0$$

$$\frac{\partial L}{\partial t}=0$$

Chain Rule: $\frac{dL}{dt}=\frac{\partial L}{\partial q_i}\dot{q}_i+\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i+\frac{\partial L}{\partial t}$

$$\frac{\partial L}{\partial q_i}\dot{q}_i+\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i-\frac{dL}{dt}=0$$

Euler-Lagrange: $\frac{\partial L}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i+\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i-\frac{dL}{dt}=0$$

Product Rule: $\dot{A}B+A\dot{B}=\frac{d}{dt}(AB)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i-L\right)=0$$

Conserved Quantity:

$$H=\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i-L$$

Noether's theorem associates every global continuous symmetry of a law of time evolution (= invariance of the action) with a conserved (= time independent) quantity [TM, Vol 1, Ch. 7 & 8; ENWT, Ch. 5; NNCM, Ch. 10.4]. The theorem makes use of the principle of "least" action, which is central to classical physics. In the following, we focus on particle theories (as opposed to field theories). Before stating Noether's theorem in its general form, it is instructive to go through some important special cases. We analyze invariance under (i) space translations, (ii) linear point transformations, which includes rotations as a special case, and (iii) time translations.

Space Translation. Symmetry with respect to space translations means that the action $\int_{t_1}^{t_2}L(q_i,\dot{q}_i,t)\,dt$, which encodes the law of time evolution, stays invariant under the transformation $q_i'=q_i+\lambda a_i$, where the q_i are (generalized) position coordinates, the a_i represent the shift direction, and λ is the transformation parameter controlling the amount of shift. Note that the (generalized) velocity coordinates are not affected by this transformation: $\dot{q}_i'=\dot{q}_i$. Because large transformations can be built up from small ones, it is sufficient to consider only transformations near the identity, parametrized by $\lambda=0$. For the action to be invariant with respect to λ, it is sufficient (but not necessary) for the Lagrangian $L(q_i,\dot{q}_i,t)$ to be invariant. The left part of the diagram shows how this invariance leads to the conserved quantity $p=p_ia_i$ (summation over i implied), where the $p_i=\partial L/\partial\dot{q}_i$ are known as the *canonical momenta conjugate to* q_i. A key step in this derivation is the use of the Euler-Lagrange equation (blue box), which relates the q_i and $\dot{q}_i$, such that the action becomes extremal (= stationary).

For the special case in which the Lagrangian does *not* depend on the position coordinate q_i (for a particular i), the conserved quantity is simply the corresponding canonical momentum $p_i=\partial L/\partial\dot{q}_i$. For example, the Lagrangian $L=\frac{1}{2}m\dot{q}^2$, which lacks a potential-energy term $V(q)$ and therefore describes a system without forces, yields the conserved quantity $p=m\dot{q}$, which is just the Newtonian momentum.

Linear Point Transformation. Symmetry with respect to linear point transformations means that the action stays invariant under the transformation $q_i' = S_{ij}(\lambda)q_j$ (summation over j implied), where S_{ij} represents the linear transformation matrix and λ is the transformation parameter. Note that the velocity coordinates transform like the position coordinates: $\dot{q}_i' = S_{ij}(\lambda)\dot{q}_i$. For a small transformation near the identity, which is parametrized by $\lambda = 0$, the transformation is characterized by the generator $G_{ij} = [\partial S_{ij}/\partial\lambda]_{\lambda=0}$. The center part of the diagram shows how this invariance leads to the conserved quantity $J = p_i S_{ij} q_j$, where the $p_i = \partial L/\partial \dot{q}_i$ are again the canonical momenta.

For the special case of 2D rotations described in Cartesian coordinates, the transformation $S_{ij}(\lambda)$ and the corresponding generator G_{ij} are given by the matrices

$$S(\lambda) = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}, \quad G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the conserved quantity is $J = xp_y - yp_x$, where we denoted the position coordinates $q_1 = x$ and $q_2 = y$, and the canonical momenta $p_1 = p_x$ and $p_2 = p_y$. We recognize J as the angular momentum!

Time Translation. Symmetry with respect to time translations means that the action stays invariant under the transformation $t' = t + \lambda$, where λ is the transformation parameter controlling the amount of time shift. We apply this shift only to the *explicit time dependence* of the Lagrangian because it is the *form* of the Lagrangian that needs to be time-translation invariant. (If we were to shift the implicit time dependence of the dynamical variables $q_i(t)$ and $\dot{q}_i(t)$ as well, we would simply redefine the time axis.) Time-translation symmetry occurs whenever the Lagrangian has *no* explicit time dependence [TM, Vol. 1, Ch. 8]. The right part of the diagram shows how this invariance leads to the conserved quantity $H = p_i\dot{q}_i - L$, known as the *Hamiltonian*, where the $p_i = \partial L/\partial\dot{q}_i$ are again the canonical momenta. The Hamiltonian represents the total energy of the system.

For example, given the Lagrangian $L = \frac{1}{2}m\dot{q}^2 - V(q)$, the conserved quantity is $H = (m\dot{q})\dot{q} - L = \frac{1}{2}m\dot{q}^2 + V(q)$, which is the sum of the kinetic and potential energy and thus represents the total energy.

Combinations and Generalizations. Symmetry under the transformation $q_i' = F_i(\lambda, q_j, t)$, where the F_i are arbitrary (differentiable) functions and $\lambda = 0$ parametrizes the identity, generalizes the above symmetries. Now, the action based on the Lagrangian $L[F_i(\lambda, q_j, t), \dot{F}_i(\lambda, q_j, t), t]$ stays invariant and the corresponding conserved quantity turns out to be $Q = p_i f_i(q_j, t)$, where $f_i(q_j, t) = [\partial F_i(\lambda, q_j, t)/\partial\lambda]_{\lambda=0}$ represents the vector field of displacements generated by a small transformation and the $p_i = \partial L/\partial\dot{q}_i$ are the usual canonical momenta [TM, Vol. 1, Ch. 7]. For the special case of translations, we have $F_i(\lambda, q_j, t) = q_i + \lambda a_i$ and thus $f_i(q_j, t) = a_i$ resulting in $Q = p_i a_i$; for the special case of linear point transformations, we have $F_i(\lambda, q_j, t) = S_{ij}(\lambda)q_j$ and thus $f_i(q_j, t) = G_{ij}q_j$, where $G_{ij} = [\partial S_{ij}/\partial\lambda]_{\lambda=0}$, resulting in $Q = p_i G_{ij} q_j$.

Finally, including symmetry under the transformation $t' = F_0(\lambda, q_j, t)$ further generalizes the above symmetries. Now, the action based on $L[F_i(\lambda, q_j, t), \dot{F}_i(\lambda, q_j, t), F_0(\lambda, q_j, t)]$ for $i = 1, 2, 3, \ldots$ stays invariant and the corresponding conserved quantity turns out to be $Q = p_i f_i(q_j, t) - Hf_0(q_j, t)$, where $f_0(q_j, t) = [\partial F_0(\lambda, q_j, t)/\partial\lambda]_{\lambda=0}$ and $H = p_i\dot{q}_i - L$ is the Hamiltonian [ENWT, Ch. 4-5]. For the special case of a pure time translation, we have $F_0(\lambda, q_j, t) = t + \lambda$, $F_i(\lambda, q_j, t) = q_i$ and thus $f_0(q_j, t) = 1$, $f_i(q_j, t) = 0$, resulting in $Q = H$.