9.8 Noether's Theorem for Field Theories; Space-Time Symmetries

Symmetry: Global Space-Time Translation

$$\left[\frac{d}{d\lambda}\mathcal{L}\left(\psi^\alpha(\vec{x}), \partial_\mu\psi^\alpha(\vec{x}), \vec{x} + \lambda\vec{a}\right)\right]_{\lambda=0} = 0$$

$$\frac{\partial\mathcal{L}}{\partial x^\nu}a^\nu = 0$$

Chain Rule: $\frac{d\mathcal{L}}{dx^\nu} = \frac{\partial\mathcal{L}}{\partial\psi^\alpha}\partial_\nu\psi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\partial_\nu(\partial_\mu\psi^\alpha) + \frac{\partial\mathcal{L}}{\partial x^\nu}$

$$\left[\frac{\partial\mathcal{L}}{\partial\psi^\alpha}\partial_\nu\psi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\partial_\nu(\partial_\mu\psi^\alpha) - \frac{d\mathcal{L}}{dx^\nu}\right]a^\nu = 0$$

Euler-Lagrange: $\frac{\partial\mathcal{L}}{\partial\psi^\alpha} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\right)$

$$\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\right)\partial_\nu\psi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\partial_\nu\partial_\mu\psi^\alpha - \frac{d\mathcal{L}}{dx^\nu}\right]a^\nu = 0$$

Product Rule: $(\partial_\mu A)B + A(\partial_\mu B) = \partial_\mu(AB)$

$$\left[\frac{d}{dx^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\partial_\nu\psi^\alpha\right) - \frac{d\mathcal{L}}{dx^\nu}\right]a^\nu = 0$$

Locally Conserved Currents ($\partial_\mu\mathcal{H}^\mu{}_\nu(\vec{x})a^\nu = 0$):

$$\mathcal{H}^\mu{}_\nu(\vec{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\partial_\nu\psi^\alpha - \delta^\mu_\nu\mathcal{L}$$

Symmetry: Global Lorentz Transformation

$$\left[\frac{d}{d\lambda}\mathcal{L}\left(\tilde{\Lambda}^\alpha_\beta(\lambda)\psi^\beta(\vec{x}), \tilde{\Lambda}^\alpha_\beta(\lambda)\partial_\mu\psi^\beta(\vec{x}), \Lambda(\lambda)\vec{x}\right)\right]_{\lambda=0} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi^\alpha}\tilde{\omega}^\alpha{}_\beta\psi^\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\tilde{\omega}^\alpha{}_\beta\partial_\mu\psi^\beta + \frac{\partial\mathcal{L}}{\partial x^\nu}\omega^\nu{}_\kappa x^\kappa = 0$$

Chain Rule, Euler-Lagrange, Product Rule

$$\frac{d}{dx^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\tilde{\omega}^\alpha{}_\beta\psi^\beta\right) + \left[\frac{d}{dx^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\alpha)}\partial_\nu\psi^\alpha\right) - \frac{d\mathcal{L}}{dx^\nu}\right]\omega^\nu{}_\kappa x^\kappa = 0$$

$$\frac{d}{dx^\mu}\left(\Pi^\mu{}_\alpha\tilde{\omega}^\alpha{}_\beta\psi^\beta\right) + \frac{d}{dx^\mu}\left(\mathcal{H}^\mu{}_\nu\right)\omega^\nu{}_\kappa x^\kappa = 0$$

Symmetrize: $\mathcal{H}^\mu{}_\nu \rightarrow T^{\mu\nu}$

$$\frac{d}{dx^\mu}\left(\Pi^\mu{}_\alpha\tilde{\omega}^\alpha{}_\beta\psi^\beta\right) + \frac{d}{dx^\mu}\left(T^{\mu\nu}\right)\omega_{\nu\kappa}x^\kappa = 0$$

Symmetric × Antisymmetric: $T^{\mu\nu}\omega_{\nu\kappa}\partial_\mu x^\kappa = 0$

$$\frac{d}{dx^\mu}\left(\Pi^\mu{}_\alpha\tilde{\omega}^\alpha{}_\beta\psi^\beta + T^{\mu\nu}\omega_{\nu\kappa}x^\kappa\right) = 0$$

Locally Conserved Currents ($\partial_\mu\mathcal{J}^{\mu\nu\kappa}(\vec{x})\omega_{\nu\kappa} = 0$):

$$\mathcal{J}^{\mu\nu\kappa}(\vec{x}) = \Pi^\mu{}_\alpha\Omega^{\alpha}_{\beta}{}^{\nu\kappa}\psi^\beta + T^{\mu\nu}x^\kappa$$

Continuing our discussion of Noether's theorem, we now focus on space-time symmetries (as opposed to internal symmetries). We study invariance under (i) global space-time translations and (ii) global Lorentz transformations, which combine space rotations and boosts. In the following, we write the field components as ψ^α. In the case of a 4-vector field, α is a space-time index.

Global Space-Time Translation. Symmetry with respect to global space-time translations means that the action $\int_{x_1}^{x_2}\mathcal{L}(\psi^\alpha(\vec{x}), \partial_\mu\psi^\alpha(\vec{x}), \vec{x})\, d^4x$, which encodes the law of time evolution, stays invariant under the transformation $\vec{x}' = \vec{x} + \lambda\vec{a}$, where $\vec{a}$ represents the global space-time shift direction and λ is the transformation parameter controlling the amount of shift. Rewritten in terms of components, this transformation is $(x^\mu)' = x^\mu + \lambda a^\mu$. Note that the field components are not affected by such a shift: $(\psi^\alpha)' = \psi^\alpha$. Similar to our discussion of time-translation symmetry, the space-time translation applies only to the *explicit space-time dependence* of the Lagrangian density and not to the implicit space-time dependence of the fields. If the Lagrangian density has no explicit space-time dependence, space-time translation symmetry holds for any shift direction a^μ. The left part of the diagram shows how this invariance leads to the locally conserved current $\mathcal{H}^\mu{}_\nu(\vec{x})a^\nu$ (summation over ν implied), where $\mathcal{H}^\mu{}_\nu(\vec{x}) = \Pi^\mu{}_\alpha(\vec{x})\partial_\nu\psi^\alpha(\vec{x}) - \delta^\mu_\nu\mathcal{L}(\vec{x})$ is known as the *Hamiltonian (density) tensor* [ENWT, Ch. 6.3] and $\Pi^\mu{}_\alpha(\vec{x}) = \partial\mathcal{L}/\partial(\partial_\mu\psi^\alpha(\vec{x}))$ are the canonical momentum densities, as before. The Hamiltonian tensor represents the densities and fluxes of the energy and momentum in the field. Pulling the second index of $\mathcal{H}^\mu{}_\nu$ up and symmetrizing the tensor yields the better-known *energy-momentum (density) tensor* $T^{\mu\nu}$ [QFTGA, Ch. 10.3]. A key step in this derivation is the use of the Euler-Lagrange equation (blue box), which relates the $\psi^\alpha(\vec{x})$ and $\partial_\mu\psi^\alpha(\vec{x})$, such that the action becomes extremal (= stationary).

A globally conserved 4-vector can be obtained by integrating the time-like components $\mathcal{H}^0{}_\mu(\vec{x})$ or, equivalently, $T^{0\mu}(\vec{x})$ over all of space: $p^\mu = \int T^{0\mu}d^3x$, where we assumed full space-time translation symmetry (arbitrary a^μ). This 4-vector can be broken up into one component for the total field energy

$E = \int T^{00} d^3x$ and three components for the total field momentum is $p_i = \int T^{0i} d^3x$, where $i = 1, 2, 3$, and thus is known as the *energy-momentum 4-vector*.

Given the Klein-Gordon Lagrangian density $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2$, the canonical momentum densities are $\Pi^\mu = \partial^\mu\phi$. Plugging this into the formula for the Hamiltonian tensor, $\mathcal{H}^\mu_{\ \nu} = \Pi^\mu\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$, we find $\mathcal{H}^\mu_{\ \nu} = \partial^\mu\phi\partial_\nu\phi - \frac{1}{2}\delta^\mu_\nu(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2)$, which evaluates to

$$(\mathcal{H}^\mu_{\ \nu}) = \begin{pmatrix} \mathcal{H} & \partial_t\phi\partial_x\phi & \partial_t\phi\partial_y\phi & \partial_t\phi\partial_z\phi \\ -\partial_x\phi\partial_t\phi & \mathcal{H} - (\partial_t\phi)^2 - (\partial_x\phi)^2 & -\partial_x\phi\partial_y\phi & -\partial_x\phi\partial_z\phi \\ -\partial_y\phi\partial_t\phi & -\partial_y\phi\partial_x\phi & \mathcal{H} - (\partial_t\phi)^2 - (\partial_y\phi)^2 & -\partial_y\phi\partial_z\phi \\ -\partial_z\phi\partial_t\phi & -\partial_z\phi\partial_x\phi & -\partial_z\phi\partial_y\phi & \mathcal{H} - (\partial_t\phi)^2 - (\partial_z\phi)^2 \end{pmatrix},$$

where $\mathcal{H} = \mathcal{H}^0_{\ 0} = \frac{1}{2}[(\partial_t\phi)^2 + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2 + m^2\phi^2]$ is the *Hamiltonian density* (= energy density of the field). The total field energy is $E = \int \mathcal{H} d^3x$ and the total field momentum in the x direction is $p_x = \int \partial_t\phi\partial_x\phi \, d^3x$, etc., all of which are globally conserved.

Global Lorentz Transformation. Symmetry with respect to global Lorentz transformations means that the action stays invariant under the combined transformations of the space-time coordinates $(x^\mu)' = \Lambda^\mu_{\ \nu}(\lambda)x^\nu$ and the field components $(\psi^\alpha)' = \widetilde{\Lambda}^\alpha_{\ \beta}(\lambda)\psi^\beta$, where $\Lambda^\mu_{\ \nu}(\lambda)$ is the 4-vector representation of the Lorentz group and $\widetilde{\Lambda}^\alpha_{\ \beta}(\lambda)$ is the appropriate representation for the field type in question (e.g., 4-vector, spinor, or scalar field). For a small transformation near the identity (parametrized by $\lambda = 0$), the transformations are characterized by the generators $\omega^\mu_{\ \nu} = [\partial\Lambda^\mu_{\ \nu}/\partial\lambda]_{\lambda=0}$ and $\widetilde{\omega}^\alpha_{\ \beta} = [\partial\widetilde{\Lambda}^\alpha_{\ \beta}/\partial\lambda]_{\lambda=0}$. These generators are linearly related like $\widetilde{\omega}^\alpha_{\ \beta} = \Omega^{\alpha\ \ \nu}_{\ \beta\mu}\omega^\mu_{\ \nu}$: for 4-vector fields $\widetilde{\omega}^\alpha_{\ \beta} = \delta^\alpha_\mu\delta^\nu_\beta\omega^\mu_{\ \nu}$, for Dirac-spinor fields $\widetilde{\omega}^\alpha_{\ \beta} = \frac{1}{4}[\gamma^\mu]^\alpha_{\ \delta}[\gamma^\nu]^\delta_{\ \beta}\omega_{\mu\nu}$, and for scalar fields $\widetilde{\omega}^\alpha_{\ \beta} = 0$. The right part of the diagram shows how this invariance leads to the locally conserved current $\mathcal{J}^{\mu\nu\kappa}(\vec{x})\omega_{\nu\kappa}$, where $\mathcal{J}^{\mu\nu\kappa}(\vec{x}) = \Pi^\mu_{\ \alpha}(\vec{x})\Omega^{\alpha\ \nu\kappa}_{\ \beta}\psi^\beta(\vec{x}) + T^{\mu\nu}(\vec{x})x^\kappa$, $\Pi^\mu_{\ \alpha}(\vec{x})$ are the canonical momentum densities, and $T^{\mu\nu}(\vec{x})$ is the energy-momentum tensor, as before. The derivation starts out as usual: chain rule, Euler-Lagrange, and product rule (only the result is shown in the diagram). Then we take advantage of the fact that the energy-momentum tensor, $T^{\mu\nu}$, is symmetric and the generator of the Lorentz transformation with both indices lowered, $\omega_{\nu\kappa}$, is antisymmetric to extend the ∂_μ operator across the product $T^{\mu\nu}\omega_{\nu\kappa}x^\kappa$ [QFTGA, Ch. 10.3]. Finally, using the antisymmetry of $\omega_{\nu\kappa}$ again, we can antisymmetrize $\mathcal{J}^{\mu\nu\kappa}$ without loss of generality: $\tilde{\mathcal{J}}^{\mu\nu\kappa} = \mathcal{J}^{\mu[\nu\kappa]} := \frac{1}{2}(\mathcal{J}^{\mu\nu\kappa} - \mathcal{J}^{\mu\kappa\nu})$. For vector and spinor fields, the conserved currents have two parts: one due to the transformation of the field components, $\Pi^\mu_{\ \alpha}\Omega^{\alpha\ [\nu\kappa]}_{\ \beta}\psi^\beta$, and another one due to the transformation of the space-time coordinates, $T^{\mu[\nu}x^{\kappa]}$. For scalar fields the first part is zero.

A globally conserved tensor can be obtained by integrating the time-like components $\tilde{\mathcal{J}}^{0\mu\nu}(\vec{x})$ over all of space: $M^{\mu\nu} = \int \tilde{\mathcal{J}}^{0\mu\nu} d^3x$, where we assumed full Lorentz symmetry (arbitrary $\omega_{\mu\nu}$). This antisymmetric tensor can be broken up into three components for the total angular momentum of the field $J_i = \frac{1}{2}\varepsilon_{ijk}\int \tilde{\mathcal{J}}^{0jk} d^3x$ and three components for the total center-of-mass motion of the field $N_i = \int \tilde{\mathcal{J}}^{00i} d^3x$, where $i = 1, 2, 3$, and thus is known as the *angular-momentum tensor* (or *6-angular momentum* [RtR, Ch. 18.7]). The angular momentum due to the transformation of the field components is known as *intrinsic angular momentum* or *spin* and the part due to the transformation of the space-time coordinates is known as *orbital angular momentum*, $J_i = S_i + L_i$.