9.7 Noether's Theorem for Field Theories; Internal Symmetries

Symmetry: Global Field Shift

$$\left[\frac{d}{d\lambda}\mathcal{L}\big(\psi_i(\vec{x}) + \lambda\alpha_i, \partial_\mu\psi_i(\vec{x}), \vec{x}\big)\right]_{\lambda=0} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi_i}\alpha_i = 0$$

Euler-Lagrange: $\frac{\partial\mathcal{L}}{\partial\psi_i} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)\alpha_i = 0$$

$$\Downarrow$$

Locally Conserved Current ($\partial_\mu\Pi^\mu(\vec{x}) = 0$):

$$\Pi^\mu(\vec{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\alpha_i$$

Globally Conserved Quantity ($dP/dt = 0$):

$$P = \int_{V\to\infty}\Pi^0\,(\vec{x})\,d^3x$$

Symmetry: Global Linear Field Transformation

$$\left[\frac{d}{d\lambda}\mathcal{L}\big(S_{ij}(\lambda)\psi_j(\vec{x}), S_{ij}(\lambda)\partial_\mu\psi_j(\vec{x}), \vec{x}\big)\right]_{\lambda=0} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi_i}G_{ij}\psi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}G_{ij}\partial_\mu\psi_j = 0$$

Euler-Lagrange: $\frac{\partial\mathcal{L}}{\partial\psi_i} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)G_{ij}\psi_j + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}G_{ij}\partial_\mu\psi_j = 0$$

Product Rule: $(\partial_\mu A)B + A(\partial_\mu B) = \partial_\mu(AB)$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}G_{ij}\psi_j\right) = 0$$

Locally Conserved Current ($\partial_\mu J^\mu(\vec{x}) = 0$):

$$J^\mu(\vec{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}G_{ij}\psi_j$$

Globally Conserved Quantity ($dQ/dt = 0$):

$$Q = \int_{V\to\infty}J^0\,(\vec{x})\,d^3x$$

To get from a particle theory to a field theory, we need the following changes [TM, Vol 3]:

- The dynamical variables (= degrees of freedom) change from the (generalized) position coordinates q_i to the *field components* ψ_i.
- The dynamical variables change from functions of time, $q(t)$, to functions of *space-time*, $\psi(t,\vec{x}) = \psi(\vec{x})$, where $\vec{x} = (t,x,y,z)^T$. The position coordinates $\vec{x}$ change from dynamical variables to just labels (arguments of the field function). The derivatives change from time derivatives, $\dot{q} = \partial_t q$, to space-time derivatives: $\partial_\mu\psi$.
- The Lagrangian L changes to the *Lagrangian density* $\mathcal{L}$, where $L = \int\mathcal{L}\,d^3x$.

Noether's theorem for field theories associates every global continuous symmetry of a law of time evolution (= invariance of the action) with a locally conserved current (= continuity equation). In the following, we focus on internal symmetries (as opposed to space-time symmetries), which are easier to understand and more closely aligned with the symmetries of particle theories. We study invariance under (i) global field shifts and (ii) global linear field transformations.

Global Field Shift. Symmetry with respect to global field shifts means that the action $\int_{x_1}^{x_2}\mathcal{L}(\psi_i(\vec{x}), \partial_\mu\psi_i(\vec{x}), \vec{x})\,d^4x$, which encodes the law of time evolution, stays invariant under the transformation $\psi_i'(\vec{x}) = \psi_i(\vec{x}) + \lambda\alpha_i$, where ψ_i are the field components, α_i represents the global shift direction in the (internal) field space, and λ is the transformation parameter controlling the amount of shift. Note that the field derivatives are not affected by this transformation: $(\partial_\mu\psi_i(\vec{x}))' = \partial_\mu\psi_i(\vec{x})$. Because large transformations can be built up from small ones, it is sufficient to consider only transformations near the identity, parametrized by $\lambda = 0$. For the action to be invariant with respect to λ, it is sufficient (but not necessary) for the Lagrangian density $\mathcal{L}(\psi_i(\vec{x}), \partial_\mu\psi_i(\vec{x}), \vec{x})$ to be invariant. The left part of the diagram shows how this invariance leads to the locally conserved current $\Pi^\mu(\vec{x}) =$

$\Pi^{\mu}_{\ i}(\vec{x})a_i$ (summation over i implied), where the $\Pi^{\mu}_{\ i}(\vec{x}) = \partial\mathcal{L}/\partial(\partial_\mu\psi_i(\vec{x}))$ are known as the *canonical momentum densities* conjugate to the fields $\psi_i(\vec{x})$. A key step in this derivation is the use of the Euler-Lagrange equation (blue box), which relates the $\psi_i(\vec{x})$ and $\partial_\mu\psi_i(\vec{x})$, such that the action becomes extremal (= stationary).

The 4-divergence of the current $\Pi^\mu(\vec{x})$ is zero: $\partial_\mu\Pi^\mu(\vec{x}) = 0$. This amounts to a *continuity equation,* which expresses *local conservation,* as can be seen by breaking the 4-vector field into a time-like field, $\pi(\vec{x}) = \Pi^0(\vec{x})$, and a space-like field, $\vec{j}(\vec{x}) = \Pi^k(\vec{x})$ with $k = 1, 2, 3$, and writing $d/dt\, \pi(\vec{x}) = \vec{\nabla}\cdot\vec{j}(\vec{x})$. Integrating this continuity equation over all of space, applying Gauss' theorem (second equal sign shown below), and assuming that $\vec{j}(x)$ vanishes at infinity (third equal sign) yields zero,

$$\frac{d}{dt}\int_{V\to\infty}\pi(\vec{x})\, d^3x = \int_{V\to\infty}\vec{\nabla}\cdot\vec{j}(\vec{x})\, d^3x = \int_{S\to\infty}\vec{j}(\vec{x})\cdot\vec{n}\, d^2x = 0,$$

revealing that the integrated time-like momentum density is a *globally conserved* quantity.

The time-like momentum density, $\pi(\vec{x})$, plays an important role in *quantum field theory* (QFT) [QFTGA, Ch. 11; PfS, Ch. 5.2; NNQFT, Ch. 8.2]. Just like we are able to upgrade classical mechanics to quantum mechanics by imposing the commutation relations $[q_i, p_j] = i\hbar\delta_{ij}$ (turning the variables p_i into operators), we can upgrade classical field theory to QFT by imposing the commutation relations $[\psi_i(\vec{x}), \pi_j(\vec{y})] = i\delta(\vec{x}-\vec{y})\delta_{ij}$, at least for bosonic fields (making the fields $\pi_i(\vec{x})$ operator valued).

Global Linear Field Transformation. Symmetry with respect to global linear field transformations means that the action stays invariant under the transformation $\psi_i'(\vec{x}) = S_{ij}(\lambda)\psi_j(\vec{x})$, where S_{ij} represents the global linear transformation matrix and λ is the transformation parameter. Note that the field derivatives transform like the field itself: $(\partial_\mu\psi_i(\vec{x}))' = S_{ij}(\lambda)\partial_\mu\psi_j(\vec{x})$. For a small transformation near the identity (parametrized by $\lambda = 0$), the transformation is characterized by the generator $G_{ij} = [\partial S_{ij}/\partial\lambda]_{\lambda=0}$. The right part of the diagram shows how this invariance leads to the locally conserved current $J^\mu(\vec{x}) = \Pi^{\mu}_{\ i}(\vec{x})G_{ij}\psi_j(\vec{x})$, where the $\Pi^{\mu}_{\ i}(\vec{x}) = \partial\mathcal{L}/\partial(\partial_\mu\psi_i(\vec{x}))$ are again the canonical momentum densities. A globally conserved quantity, Q, is obtained by integrating the time-like component $J^0(\vec{x})$ over all of space. The locally conserved current, $J^\mu(\vec{x})$, is sometimes referred to as a *Noether current* and the globally conserved quantity, Q, as a *Noether charge*.

For the complex field $\psi(\vec{x})$ with global U(1) symmetry, we can identify $\psi_1 = \psi$ and $\psi_2 = \bar{\psi}$,

$$S(\lambda) = \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix}, \quad G = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus, the locally conserved current is $J^\mu(\vec{x}) = i\Pi^{\mu}_{\ 1}(\vec{x})\psi(\vec{x}) - i\Pi^{\mu}_{\ 2}(\vec{x})\bar{\psi}(\vec{x})$. Given the Dirac Lagrangian density, we find $\Pi^{\mu}_{\ 1} = \partial\mathcal{L}/\partial(\partial_\mu\psi) = i\bar{\psi}\gamma^\mu$ and $\Pi^{\mu}_{\ 2} = \partial\mathcal{L}/\partial(\partial_\mu\bar{\psi}) = 0$ and hence $J^\mu = -\bar{\psi}\gamma^\mu\psi$. Multiplying this current by the elementary charge $-q$, yields the *electric 4-current* $J^\mu_{\text{elec}} = q\bar{\psi}\gamma^\mu\psi$ of a charged Dirac spinor field. Integrating the time-like component of this electric current over all of space yields the conserved *electric charge* $Q_{\text{elec}} = q\int\psi^\dagger\psi d^3x$ [QFTGA, Ch. 38.2; PfS, Ch. 7.1.6].

Similarly, a weakly interacting doublet field with global SU(2) symmetry implies the conservation of (weak) isospin [PfS, Ch. 7.7] and a strongly interacting triplet field with global SU(3) symmetry implies the conservation of color charge [PfS, Ch. 7.8.1].