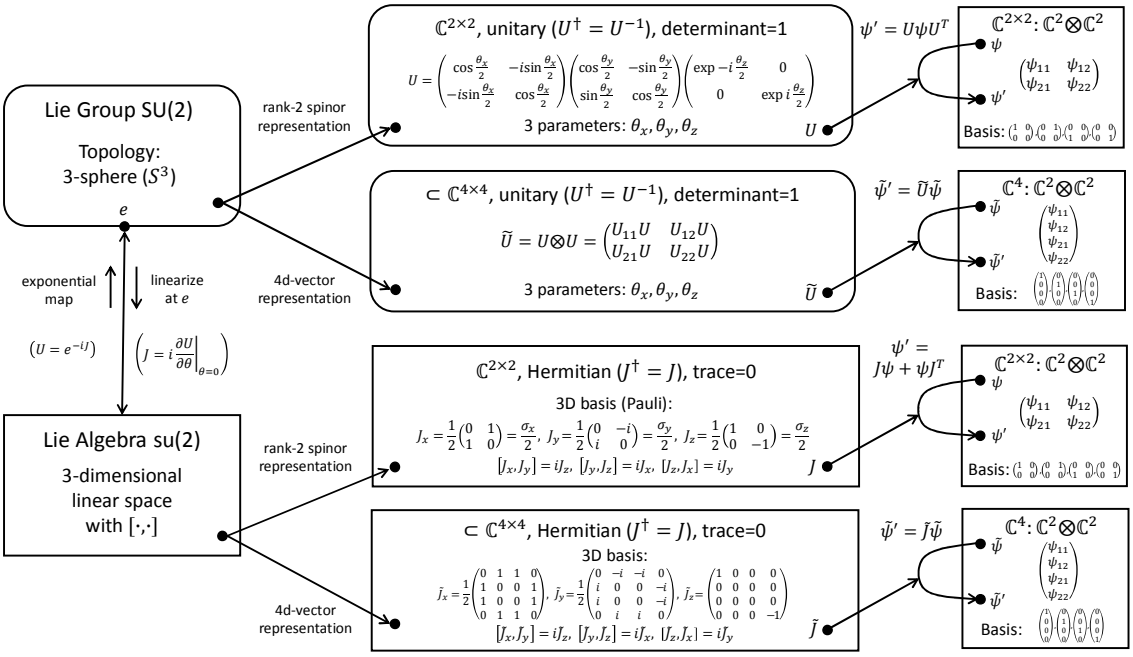
3.16 SU(2): Tensor-Product Representation on 4D Vectors; Singlet and Triplet States



To bring the tensor-product representation from the previous example into a more familiar form, we "flatten" the rank-2 spinor from its 2×2-matrix form into a 4-component vector, that is, we map $\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$ to $\tilde{\psi} = (\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})^T$. Now, we have to find out how the Lie-group elements act on this column vector. It turns out that the sandwich operation $\psi' = U\psi U^T$ from before maps to the straightforward matrix-vector product $\tilde{\psi}' = \widetilde{U}\tilde{\psi}$, where $\widetilde{U}$ is now a 4×4 matrix. This new matrix is related to U by the *Kronecker product* as follows: $\widetilde{U} = U\otimes U = \begin{pmatrix} U_{11}U & U_{12}U \\ U_{21}U & U_{22}U \end{pmatrix}$. Note that each element of the 2×2 matrix on the right-hand side is a 2×2 matrix, that is, the whole object is a 4×4 matrix. (For more on the Kronecker product, see [TM, Vol. II, Ch.7].)

Next, we have to find out how the Lie-algebra elements act on the column vector. It turns out that the two-part operation $\psi' = J\psi + \psi J^T$ from before maps to the straightforward matrix-vector product $\tilde{\psi}' = \tilde{J}\tilde{\psi}$, where $\tilde{J}$ is now a 4×4 matrix as well. To find $\tilde{J}$, we differentiate $\widetilde{U}$ with respect to a parameter and evaluate the result at the identity element, which yields $\tilde{J} = J\otimes I + I\otimes J = \begin{pmatrix} J_{11}I & J_{12}I \\ J_{21}I & J_{22}I \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, where I is the 2×2 identity matrix and 0 is the 2×2 zero matrix. Knowing the basis generators J_x, J_y, J_z of the defining representation, it is easy to calculate the basis generators $\tilde{J}_x, \tilde{J}_y, \tilde{J}_z$ of our new 4-dimensional representation (see the lower branch of the diagram).

Now that we are back on familiar territory, we can calculate the eigenvectors of $\tilde{J}_z$ and construct a meaningful basis for our new 4-dimensional representation space. Solving the (trivial) eigenequation $\tilde{J}_z\Psi_m = m\Psi_m$, we find the eigenvector $\Psi_{+1} = (1, 0, 0, 0)^T$ with eigenvalue $m = +1$, the eigenvector $\Psi_{-1} = (0, 0, 0, 1)^T$ with eigenvalue $m = -1$, and the eigen*space* $\Psi_0 = (0, \alpha, \beta, 0)^T$ with eigenvalue $m = 0$. The last two eigenvectors can be chosen arbitrarily within the 2-dimensional space (= plane) given by Ψ_0. So, just like in the case of the infinite-dimensional representation on functions, the

operator $\tilde{J}_z$ alone can't give us a unique basis. Again, we need a second operator that commutes with $\tilde{J}_z$, such as the Casimir operator $J_C^2 = \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2$, which we are already familiar with. Plugging in the basis generators, we find $J_C^2 = \sum_{i=1}^{3}(J_i \otimes I + I \otimes J_i)^2 = \frac{1}{2}(3I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Solving the simultaneous eigenequations $\tilde{J}_z \Psi_{j,m} = m\Psi_{j,m}$ and $J_C^2 \Psi_{j,m} = j(j+1)\Psi_{j,m}$, we find the four joint eigenvectors $\Psi_{0,0} = 1/\sqrt{2} \cdot (0, 1, -1, 0)^T$, $\Psi_{1,0} = 1/\sqrt{2} \cdot (0, 1, 1, 0)^T$, $\Psi_{1,+1} = (1, 0, 0, 0)^T$, and $\Psi_{1,-1} = (0, 0, 0, 1)^T$, all of which are now unique. The eigenspace with $m = 0$ is now spanned by two eigenvectors, one with $j = 0$ and one with $j = 1$.

Is this 4-dimensional representation reducible? Yes, it is: the eigenvalue, $j(j+1)$, of the Casimir operator delineates the irreducible representations. The representation breaks up into a 1-dimensional representation acting on $\Psi_{0,0}$ ($j = 0$) and a 3-dimensional representation acting on the space spanned by $\Psi_{1,+1}$, $\Psi_{1,0}$, and $\Psi_{1,-1}$ ($j = 1$).

In quantum mechanics, the four eigenvectors are states of definite combined spin. The (squared) total spin is given by the eigenvalue $j(j+1)$ and the spin along the z axis is given by the eigenvalue m. How are we to interpret these four eigenstates? The state $\Psi_{1,+1} = (1, 0, 0, 0)^T$ represents both particles with "spin up", that is, each spin is +½ (along the z axis), resulting in a combined spin of +1 (along the same axis). Similarly, the state $\Psi_{1,-1} = (0, 0, 0, 1)^T$ represents both particles with "spin down" (−½), resulting in a combined spin of −1. In both of these states, the particles are not entangled. In contrast, the remaining two states $\Psi_{0,0} = 1/\sqrt{2} \cdot (0, 1, -1, 0)^T$ and $\Psi_{1,0} = 1/\sqrt{2} \cdot (0, 1, 1, 0)^T$ represent *entangled* particles. When measured along the z axis, one particle comes out as up (spin +½) and the other one as down (spin −½), resulting in a combined spin of zero. Whereas the measured spin of each particle is random, the combined spin (along the z axis) is well defined and always zero.

The $\Psi_{0,0}$ state is called a *singlet* state and a state in the space spanned by $\Psi_{1,+1}$, $\Psi_{1,0}$, $\Psi_{1,-1}$ is called a *triplet* state. The singlet state is characterized by the combined total spin $j = 0$. This state doesn't change under rotation and the two particles are always maximally entangled. If separated and measured, the two particles exhibit "spooky action at a distance". A triplet state is characterized by the combined total spin $j = 1$. The components of this state change under rotation and so does the amount of entanglement between the two particles. Entanglement is maximal for a pure $\Psi_{1,0}$ state. (See [TM, Vol. II, Ch. 6-7] for more on singlet and triplet states.)

A good example for a system consisting of two spin-½ particles is the hydrogen atom. One particle is the proton in the nucleus and the other particle is the electron (which we assume to be in its ground-state orbital). The combined system can be in any superposition of the singlet and a triplet state. Because the two charged and spinning particles have a magnetic moment, the energy of the singlet state differs slightly from that of a triplet state. (All the triplet states must have the same energy due to rotational symmetry.) This effect is known as the *hyperfine splitting* of the energy levels in the hydrogen atom [FLP, Vol. III, Ch. 12]. More specifically, the energy of the singlet state is *lower* than that of a triplet state by h × (1420 GHz). If we wait long enough, the hydrogen atom "decays" into the lower-energy, maximally-entangled singlet state. (Note that the energy operator [= Hamiltonian] for this system, $H \propto \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$ [FLP, Vol. III, Ch. 12], is closely related to the Casimir operator.)